George E. Andrews
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Ramanujan's Lost Notebook

Part IV

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Appearing in this photograph are the participants at the second meeting of the Indian Mathematical Society on 11–13 January 1919 in Bombay. Several people important in the life of Ramanujan are pictured. Sitting on the ground are (third from left) S. Narayana Aiyar, Chief Accountant of the Madras Port Trust Office, and (fourth from left) P.V. Seshu Aiyar, Ramanujan's mathematics instructor at the Government College of Kumbakonam. Sitting in the chairs are (fifth from left) V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, and (third from right) R. Ramachandra Rao, who provided a stipend for Ramanujan for 15 months. Standing in the third row is (second from left) S.R. Ranganathan, who wrote the first book-length biography of Ramanujan in English. Identifications of the remainder of the delegates in the photograph may be found in Volume 11 of the *Journal of the Indian Mathematical Society* or [65, p. 27].

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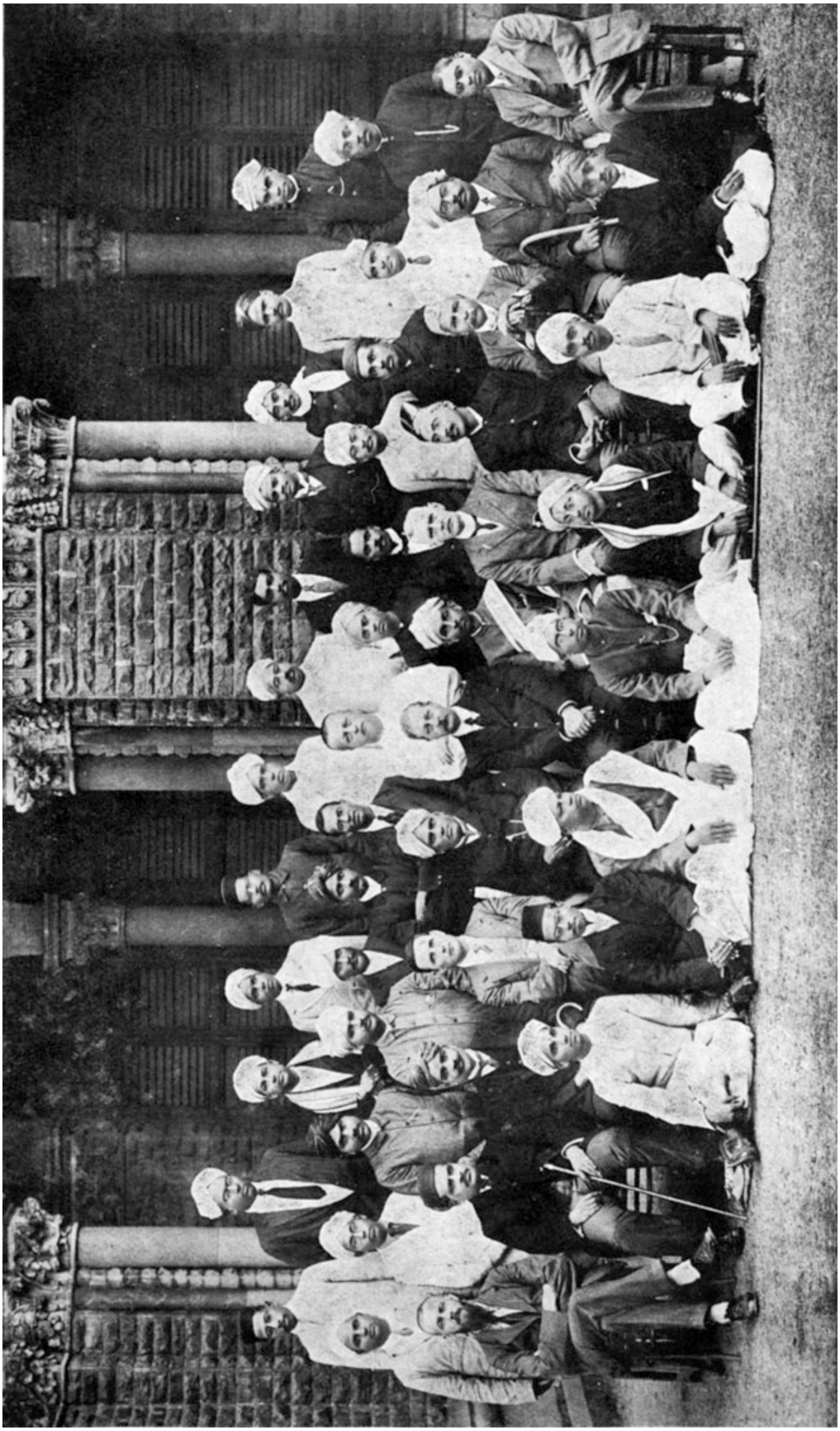

It was not until today that I discovered at last what I had been so long searching for. The treasure hidden here is greater than that of the richest king in the world and to find it, the meaning of only one more sign had to be deciphered.

—Rabindranath Tagore, “The Hidden Treasure”

Preface

This is the fourth of five volumes that the authors are writing in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*. Published by Narosa in 1988, the treatise contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. Although some of the claims examined in our fourth volume are found in the original lost notebook, most of the claims examined here are from the partial manuscripts and fragments. Classical analysis and classical analytic number theory are featured.

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Contents

1

Introduction

In contrast to our first three volumes [12–14] devoted to *Ramanujan's Lost Notebook and Other Unpublished Papers* [269], this volume does not focus on q-series. Number theory and classical analysis are in the spotlight in the present book, which is the fourth of five projected volumes, wherein the authors plan to discuss all the claims made by Ramanujan in [269]. As in our previous volumes, in the sequel, we liberally interpret *lost notebook* not only to include the original lost notebook found by the first author in the library at Trinity College, Cambridge, in March 1976, but also to include all of the material published in [269]. This includes letters that Ramanujan wrote to G.H. Hardy from nursing homes, several partial manuscripts, and miscellaneous papers. Some of these manuscripts are located at Oxford University, are in the handwriting of G.N. Watson, and are "copied from loose papers." However, it should be emphasized that the original manuscripts in Ramanujan's handwriting can be found at Trinity College Library, Cambridge.

We now relate some of the highlights in this volume, while at the same time offering our thanks to several mathematicians who helped prove some of these results.

Chapter 2 is devoted to two intriguing identities involving double series of Bessel functions found on page 335 of [269]. One is connected with the classical *circle problem*, while the other is conjoined to the equally famous *Dirichlet divisor problem*. The double series converge very slowly, and the identities were extremely difficult to prove. Initially, the second author and his collaborators, Sun Kim and Alexandru Zaharescu, were not able to prove the identities with the order of summation as prescribed by Ramanujan, i.e., the identities were proved with the order of summation reversed [57, 71]. It is possible that Ramanujan intended that the summation indices should tend to infinity "together." The three authors therefore also proved the two identities with the product of the summation indices tending to ∞ [57]. Finally, these authors proved Ramanujan's first identity with the order of summation as prescribed by Ramanujan [60]. It might be remarked here that the proofs under the three interpretations of the summation indices are entirely different; the authors

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did not use any idea from one proof in the proofs of the same identity under different interpretations. In Chap. 2, we provide proofs of the two identities with the order of summation indicated by Ramanujan in the first identity and with the order of summation reversed in the second identity. We also establish the identities when the product of the two indices of summation tends to infinity. In addition to thanking Sun Kim and Alexandru Zaharescu for their collaborations, the present authors also thank O-Yeat Chan, who performed several calculations to discern the convergence of these and related series.

It came as a huge surprise to us while examining pages in [269] when we espied famous formulas of N.S. Koshliakov and A.P. Guinand, although Ramanujan wrote them in slightly disguised forms. Moreover, we discovered that Ramanujan had found some consequences of these formulas that had not theretofore been found by any other authors. We are grateful to Yoonbok Lee and Jaebum Sohn for their collaboration on these formulas, which are the focus of Chap. 3.

Chapter 4, on the classical gamma function, features two sets of claims. We begin the chapter with some integrals involving the gamma function in the integrands. Secondly, we examine a claim that reverts to a problem [260] that Ramanujan submitted to the *Journal of the Indian Mathematical Society*, which was never completely solved. On page 339 in [269], Ramanujan offers a refinement of this problem, which was proved by the combined efforts of Ekaterina Karatsuba [177] and Horst Alzer [4].

Hypergeometric functions are featured in Chap. 5. This chapter contains two particularly interesting results. The first is an explicit representation for a quotient of two particular bilateral hypergeometric series, which was proved in a paper [50] by the second author and Wenchang Chu, whom we thank for his expert collaboration. We also appreciate correspondence with Tom Koornwinder about one particular formula on bilateral series that was crucial in our proof. Ramanujan's formula is so unexpected that no one but Ramanujan could have discovered it! The second is a beautiful continued fraction, for which Soon-Yi Kang, Sung-Geun Lim, and Sohn [175] found two entirely different proofs, each providing a different understanding of the entry. A further beautiful continued fraction of Ramanujan was only briefly examined in [175], but Kang supplied us with a very nice proof, which appears here for the first time.

Chapter 6 contains accounts of two incomplete manuscripts on Euler's constant γ, one of which was coauthored by the second author with Doug Bowman [46] and the other of which was coauthored by the second author with Tim Huber [55].

Sun Kim kindly collaborated with the second author on Chap. 7, on an unusual problem examined in a rough manuscript by Ramanujan on Diophantine approximation [56]. She also worked with the second author and Zaharescu on another partial manuscript providing the best possible Diophantine approximation to $e^{2/a}$, where a is any nonzero integer [61].

This manuscript was another huge surprise to us, for it had never been noticed by anyone, to the best of our knowledge, that Ramanujan had derived the best possible Diophantine approximation to $e^{2/a}$, which was first proved in print approximately 60 years after Ramanujan had found his proof. A third manuscript on Diophantine approximation in [269] turned out to be without substance, unless we have grossly misinterpreted Ramanujan's claims on page 343 of [269].

We next collect some results from number theory, not all of which are correct. At the beginning of Chap. 8, in Sect. 8.1, we relate that Ramanujan had anticipated the famous work of L.G. Sathe [275–278] and A. Selberg [281] on the distribution of primes, although Ramanujan did not state any specific theorems. In prime number theory, Dickman's function is a famous and useful function, but in Sect. 8.2, we see that Ramanujan had discovered Dickman's function at least 10 years before Dickman did in 1930 [106]. A.J. Hildebrand, a colleague of the second author, supplied a clever proof of Ramanujan's formula for, in standard notation, $\Psi(x, x^{\epsilon})$ and then provided us with a heuristic argument that might have been the approach used by Ramanujan. We then turn to a formula for $\zeta(\frac{1}{2})$, first given in Sect. 8 of Chap. 15 in Ramanujan's second notebook. In [269], Ramanujan offers an elegant reinterpretation of this formula, which renders an already intriguing result even more fascinating. Next, we examine a fragment on sums of powers that was very difficult to interpret; our account of this fragment is taken from a paper by D. Schultz and the second author [67]. One of the most interesting results in the chapter yields an unusual algorithm for generating solutions to Euler's diophantine equation $a^3 + b^3 = c^3 + d^3$. This result was established in different ways by Mike Hirschhorn in a series of papers [141, 158–160].

Chapter 9 is devoted to discarded fragments of manuscripts and partial manuscripts concerning the divisor functions $\sigma_k(n)$ and $d(n)$, respectively, the sum of the kth powers of the divisors of n, and the number of divisors of n. Some of this work is related to Ramanujan's paper [265]. An account of one of these fragments appeared in a paper that the second author coauthored with Prapanpong Pongsriiam [63].

In the next chapter, Chap. 10, we prove all of the results on page 196 of [269]. Two of the results evaluating certain Dirichlet series are especially interesting. A more detailed examination of these results can be found in a paper that the second author coauthored with Heng Huat Chan and Yoshio Tanigawa [47].

Chapter 11 contains some unusual old and new results on primes arranged in two rough, partial manuscripts. Ramanujan's manuscripts contain several errors, and we conjecture that this work predates his departure for England in 1914. Harold Diamond helped us enormously in both interpreting and correcting the claims made by Ramanujan in the two partial manuscripts examined in Chap. 11.

In Chap. 12, we discuss a manuscript that was either intended to be a paper by itself or, more probably, was slated to be the concluding portion of

Ramanujan's paper [263]. The results in this paper hark back to Ramanujan's early preoccupation with infinite series identities and the material in Chap. 14 of his second notebook [38, 268]. The second author had previously published an account of this manuscript [42]. Our account here includes a closer examination of two of Ramanujan's series by Johann Thiel, to whom we are very grateful for his contributions.

Perhaps the most fascinating formula found in the three manuscripts on Fourier analysis in the handwriting of Watson is a transformation formula involving the Riemann Ξ-function and the logarithmic derivative of the gamma function in Chap. 13. We are pleased to thank Atul Dixit, who collaborated with the second author on several proofs of this formula. One of the hallmarks of Ramanujan's mathematics is that it frequently generates further interesting mathematics, and this formula is no exception. In a series of papers [108–111], Dixit found analogues of this formula and found new bonds with the Ξ-function, in particular, with the beautiful formulas of Guinand and Koshliakov.

The second of the aforementioned manuscripts features integrals that possess transformation formulas like those satisfied by theta functions. Two of the integrals were examined by Ramanujan in two papers [256, 258], [267, pp. 59–67, 202–207], where he considered the integrals to be analogues of Gauss sums, a view that we corroborate in Chap. 14. One of the integrals, to which page 198 of [269] is devoted, was not examined earlier by Ramanujan. Ping Xu and the second author established Ramanujan's claims for this integral in [69]; the account given in Chap. 14 is slightly improved in places over that in [69]. (The authors are grateful to Noam Elkies for a historical note at the end of Sect. 14.1.)

In the third manuscript, on Fourier analysis, which we discuss in Chap. 15, Ramanujan considers some problems on Mellin transforms.

The next three chapters pertain to some of Ramanujan's earlier published papers. We then consider miscellaneous collections of results in classical analysis and elementary mathematics in the next two chapters.

Chapter 21 is devoted to some strange, partially incorrect claims of Ramanujan that likely originate from an early part of his career.

In summary, the second author is exceedingly obliged to his coauthors Doug Bowman, O.-Yeat Chan, Wenchang Chu, Atul Dixit, Tim Huber, Sun Kim, Yoonbok Lee, Sung-Geun Lim, Prapanpong Pongsriiam, Dan Schultz, Jaebum Sohn, Ping Xu, and Alexandru Zaharescu for their contributions.

As with earlier volumes, Jaebum Sohn carefully read several chapters and offered many corrections and helpful comments, for which we are especially grateful. Michael Somos offered several proofs for Chap. 20 and numerous corrections in several other chapters. Mike Hirschhorn also contributed many useful remarks.

We offer our gratitude to Harold Diamond, Andrew Granville, A.J. Hildebrand, Pieter Moree, Kannan Soundararajan, Gérald Tenenbaum, and Robert Vaughan for their comments that greatly enhanced our discussion of the

Sathe–Selberg results and Dickman's function in Sects. 8.1 and 8.2, respectively. Atul Dixit uncovered several papers by Guinand and Koshliakov of which the authors had not previously been aware. Most of Sect. 8.7 was kindly supplied to us by Jean-Louis Nicolas, with M.Tip Phaovibul also providing valuable insights. Useful correspondence with Ron Evans about Mersenne primes is greatly appreciated.

We offer our sincere thanks to Springer's TEX experts, Suresh Kumar and Rajiv Monsurate, for much technical advice, and to Springer copy editor David Kramer for several corrections and helpful remarks.

We thank the Master and Fellows of Trinity College, Cambridge, for providing the authors with the manuscripts to which this book is devoted.

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2

Double Series of Bessel Functions and the Circle and Divisor Problems

2.1 Introduction

In this chapter we establish identities that express certain finite trigonometric sums as double series of Bessel functions. These results, stated in Entries 2.1.1 and 2.1.2 below, are identities claimed by Ramanujan on page 335 in his lost notebook [269], for which no indications of proofs are given. (Technically, page 335 is *not* in Ramanujan's lost notebook; this page is a fragment published by Narosa *with* the original lost notebook.) As we shall see in the sequel, the identities are intimately connected with the famous *circle* and *divisor problems*, respectively. The first identity involves the ordinary Bessel function $J_1(z)$, where the more general ordinary Bessel function $J_\nu(z)$ is defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \tag{2.1.1}$$

The second identity involves the Bessel function of the second kind $Y_1(z)$ [314, p. 64, Eq. (1)], with $Y_\nu(z)$ more generally defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \tag{2.1.2}$$

and the modified Bessel function $K_1(z)$, with $K_\nu(z)$ [314, p. 78, Eq. (6)] defined, for $-\pi < \arg z < \frac{1}{2}\pi$, by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2}J_{-\nu}(iz) - e^{-\pi i\nu/2}J_\nu(iz)}{\sin(\nu\pi)}. \tag{2.1.3}$$

If ν is an integer n, then it is understood that we define the functions by taking the limits as $\nu \to n$ in (2.1.2) and (2.1.3).

To state Ramanujan's claims, we need to next define

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$$F(x)=\begin{cases}[x], & \text{if } x \text{ is not an integer},\\ x-\frac{1}{2}, & \text{if } x \text{ is an integer},\end{cases} \tag{2.1.4}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Entry 2.1.1 (p. 335). *Let* $F(x)$ *be defined by* (2.1.4). *If* $0<\theta<1$ *and* $x>0$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta)=\pi x\left(\frac{1}{2}-\theta\right)-\frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}-\frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{2.1.5}$$

Entry 2.1.2 (p. 335). *Let* $F(x)$ *be defined by* (2.1.4). *Then, for* $x>0$ *and* $0<\theta<1$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta)=\frac{1}{4}-x\log(2\sin(\pi\theta))$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}, \tag{2.1.6}$$

where

$$I_\nu(z):=-Y_\nu(z)-\frac{2}{\pi}K_\nu(z). \tag{2.1.7}$$

Ramanujan's formulation of (2.1.5) is given in the form

$$\left[\frac{x}{1}\right]\sin(2\pi\theta)+\left[\frac{x}{2}\right]\sin(4\pi\theta)+\left[\frac{x}{3}\right]\sin(6\pi\theta)+\left[\frac{x}{4}\right]\sin(8\pi\theta)+\cdots$$
$$=\pi x\left(\frac{1}{2}-\theta\right)-\frac{1}{4}\cot(\pi\theta)+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m\theta x})}{\sqrt{m\theta}}-\frac{J_1(4\pi\sqrt{m(1-\theta)x})}{\sqrt{m(1-\theta)}}\right.$$
$$\left.+\frac{J_1(4\pi\sqrt{m(1+\theta)x})}{\sqrt{m(1+\theta)}}-\frac{J_1(4\pi\sqrt{m(2-\theta)x})}{\sqrt{m(2-\theta)}}+\frac{J_1(4\pi\sqrt{m(2+\theta)x})}{\sqrt{m(2+\theta)}}-\cdots\right\}, \tag{2.1.8}$$

"where $[x]$ denotes the greatest integer in x if x is not an integer and $x-\frac{1}{2}$ if x is an integer." His formulation of (2.1.6) is similar. Since Ramanujan employed the notation $[x]$ in a nonstandard fashion, we think it is advisable to introduce the alternative notation (2.1.4). As we shall see in the sequel, there is some evidence that Ramanujan did not intend the double sums to be interpreted as iterated sums, but as double sums in which the product mn of the summation indices tends to ∞.

Note that the series on the left-hand sides of (2.1.5) and (2.1.6) are finite, and discontinuous if x is an integer. To examine the right-hand side of (2.1.5), we recall [314, p. 199] that, as $x \to \infty$,

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O\left(\frac{1}{x^{3/2}}\right). \tag{2.1.9}$$

Hence, as $m, n \to \infty$, the terms of the double series on the right-hand side of (2.1.5) are asymptotically equal to

$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}}\left(\frac{\cos\left(4\pi\sqrt{m(n+\theta)x} - \frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} - \frac{\cos\left(4\pi\sqrt{m(n+1-\theta)x} - \frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}}\right).$$

Thus, if indeed the double series on the right side of (2.1.5) does converge, it converges conditionally and not absolutely. A similar argument clearly pertains to (2.1.6).

We now discuss in detail Entry 2.1.1; our discourse will then be followed by a detailed account of Entry 2.1.2.

It is natural to ask what led Ramanujan to the double series on the right side of (2.1.5). Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Recall that the famous *circle problem* is to determine the precise order of magnitude, as $x \to \infty$, for the "error term" $P(x)$, defined by

$$\sideset{}{'}\sum_{0 \le n \le x} r_2(n) = \pi x + P(x), \tag{2.1.10}$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. Moreover, we define $r_2(0) = 1$. In [144], Hardy showed that $P(x) \neq O(x^{1/4})$, as x tends to ∞. (He actually showed a slightly stronger result.)

In 1906, W. Sierpiński [288] proved that $P(x) = O(x^{1/3})$, as $x \to \infty$. After Sierpiński's work, most efforts toward obtaining an upper bound for $P(x)$ have ultimately rested upon the identity

$$\sideset{}{'}\sum_{0 \le n \le x} r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{2.1.11}$$

(2.1.9), and methods of estimating the resulting trigonometric series. Here, the prime $'$ on the summation sign on the left side has the same meaning as above. The identity (2.1.11) was first published and proved in Hardy's paper [144], [150, pp. 243–263]. In a footnote, Hardy [150, p. 245] remarks, "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1)+d(2)+\cdots+d(n)$, where $d(n)$ is the number of divisors of n." Thus, it is possible that Ramanujan was the first to prove (2.1.11), although we do not know anything about his derivation.

Observe that the summands in the series on the right side of (2.1.11) are similar to those on the right side of (2.1.5). Moreover, the sums on the left side in each formula are finite sums over $n \leq x$. Thus, it seems plausible that there is a connection between these two formulas, and as we shall see, indeed there is. Ramanujan might therefore have derived (2.1.5) in anticipation of applying it to the circle problem.

In his paper [144], Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$, namely, for $a, b > 0$, [144, p. 283], [150, p. 263],

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}},$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. If we differentiate the identity above with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that for $\operatorname{Re} s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2+4\pi^2 n)^{3/2}},$$

which was the key identity in Hardy's proof that $P(x) \neq O(x^{1/4})$, as $x \to \infty$.

In summary, there is considerable evidence that while Ramanujan was at Cambridge, he and Hardy discussed the *circle problem*, and it is likely that Entry 2.1.1 was motivated by these discussions.

Note that if the factors $\sin(2\pi n\theta)$ were missing on the left side of (2.1.5), then this sum would coincide with the number of integral points (n, l) with $n, l \geq 1$ and $nl \leq x$, where the pairs (n, l) satisfying $nl = x$ are counted with weight $\frac{1}{2}$. Hence,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) = \sideset{}{'}\sum_{1 \leq n \leq x} d(n), \tag{2.1.12}$$

where $d(n)$ denotes the number of divisors of n, and the prime $'$ on the summation sign indicates that if x is an integer, only $\frac{1}{2}d(x)$ is counted. Of course, similar remarks hold for the left side of (2.1.6). Therefore one may interpret the left sides of (2.1.5) and (2.1.6) as weighted divisor sums.

Berndt and A. Zaharescu [71] first proved Entry 2.1.1, but with the order of summation on the double sum *reversed* from that recorded by Ramanujan. The authors of [71] proved this emended version of Ramanujan's claim by first replacing Entry 2.1.1 with the following equivalent theorem.

Theorem 2.1.1. *For $0 < \theta < 1$ and $x > 0$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\frac{1}{2}-\theta\right)$$
$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n+\theta}\sin^2\left(\frac{\pi(n+\theta)x}{m}\right) - \frac{1}{n+1-\theta}\sin^2\left(\frac{\pi(n+1-\theta)x}{m}\right)\right). \tag{2.1.13}$$

It should be emphasized that this reformulation fails to exist for Ramanujan's original formulation in Entry 2.1.1. After proving the aforementioned alternative version of Entry 2.1.1, the authors of [71] derived an identity involving the twisted character sums

$$d_\chi(n) = \sum_{k|n}\chi(k), \tag{2.1.14}$$

where χ is an odd primitive character modulo q. The following theorem on twisted character sums is proved in [71]; we have corrected the sign on the second expression on the right-hand side. The prime $'$ on the summation sign has the same meaning as it does in our discussions above, e.g., as in (2.1.10).

Theorem 2.1.2. *Let q be a positive integer, let χ be an odd primitive character modulo q, and let $d_\chi(n)$ be defined by (2.1.14). Then, for any $x > 0$,*

$$\sideset{}{'}\sum_{1\le n\le x} d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\bar{\chi}) + \frac{i\sqrt{x}}{\tau(\bar{\chi})}\sum_{1\le h<q/2}\bar{\chi}(h)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{h}{q})x}\right)}{\sqrt{m(n+\frac{h}{q})}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\frac{h}{q})x}\right)}{\sqrt{m(n+1-\frac{h}{q})}}\right\}, \tag{2.1.15}$$

where $L(s,\chi)$ denotes the Dirichlet L-function associated with the character χ, and $\tau(\chi)$ denotes the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e^{2\pi i m/q}. \tag{2.1.16}$$

Using Theorem 2.1.2, Berndt and Zaharescu [71] derived a representation for $\sum'_{0\le n\le x} r_2(n)$.

Corollary 2.1.1. *For any $x > 0$,*

$$\sideset{}{'}\sum_{0\le n\le x} r_2(n) = \pi x$$
$$+ 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}. \tag{2.1.17}$$

A possible advantage in using (2.1.17) in the circle problem is that $r_2(n)$ does not occur on the right side of (2.1.17), as in (2.1.11). On the other hand, the double series is likely to be more difficult to estimate than a single infinite series.

The summands in (2.1.17) have a remarkable resemblance to those in (2.1.11). It is therefore natural to ask whether the two identities are equivalent. We next show that (2.1.11) and (2.1.17) are *formally* equivalent. The key to this equivalence is a famous result of Jacobi. Let χ be the nonprincipal Dirichlet character modulo 4. Then Jacobi's formula [167], [44, p. 56, Theorem 3.2.1] is given by

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \,\mathrm{odd}}} (-1)^{(d-1)/2} =: 4d_\chi(n), \tag{2.1.18}$$

for all positive integers n. Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} r_2(k) \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4\sum_{k=1}^{\infty} \sum_{\substack{d|k \\ d \text{ odd}}} (-1)^{(d-1)/2} \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(\frac{J_1(2\pi\sqrt{m(4n+1)x})}{\sqrt{m(4n+1)}} - \frac{J_1(2\pi\sqrt{m(4n+3)x})}{\sqrt{m(4n+3)}} \right) \\
&= 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(\frac{J_1(4\pi\sqrt{m(n+\frac{1}{4})x})}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1(4\pi\sqrt{m(n+\frac{3}{4})x})}{\sqrt{m(n+\frac{3}{4})}} \right).
\end{aligned} \tag{2.1.19}$$

Hence, we have shown that (2.1.11) and (2.1.17) are versions of the same identity, *provided that the rearrangement of series in* (2.1.19) *is justified.* (J.L. Hafner [139] independently has also shown the formal equivalence of (2.1.11) and (2.1.17).)

In this chapter, we prove Entry 2.1.1 under two different interpretations, the first with the double series on the right-hand side summed in the order specified by Ramanujan, and the second with the double series on the right side interpreted as a double sum in which the product mn of the summation indices m and n tends to infinity. The former proof first appeared in a paper by Berndt, S. Kim, and Zaharescu [60], while the latter proof is taken from another paper [57] by the same trio of authors. We do not here give a proof of Entry 2.1.1 with the order of summation on the right-hand side of (2.1.5) reversed [71]. We emphasize that the three proofs of Entry 2.1.1 under different interpretations of the double sum on the right-hand side are entirely different; we are unable to use any portion or any idea of one proof in any of the other two proofs.

Having thoroughly discussed Entry 2.1.1, we turn our attention to Entry 2.1.2. Entry 2.1.2 was examined in detail in [48], where numerical calculations were extensively discussed with the conclusion that the entry might not be correct, because, in particular, the authors were not convinced that the double series of Bessel functions converges. Further evidence for the falsity of Entry 2.1.2 was also presented. Finding a proof of Entry 2.1.2, either in the form in which Ramanujan recorded it, or in the form in which the order of the double series is reversed, turned out to be more difficult than establishing a proof of Entry 2.1.2 in [71] for the following reasons: The Bessel functions $Y_1(z)$ and $K_1(z)$ have singularities at the origin. There is a lack of the "cancellation" in the pairs of Bessel functions on the right-hand side of (2.1.6) (where a plus sign separates the pairs of Bessel functions) that is evinced in (2.1.5) (where a minus sign separates the pairs of Bessel functions). We have a much less convenient intermediary theorem, Theorem 2.4.2, instead of Theorem 2.1.1, which replaces the proposed double Bessel series identity by a double trigonometric series identity. At this writing, we are unable to prove Entry 2.1.1 with the order of summation prescribed by Ramanujan. However, we can prove Entry 2.1.2 if we invert the order of summation or if we let the product of the indices of summation tend to infinity. Moreover, as we shall see in our proof, we need to make one further assumption in order to prove Entry 2.1.2 with the double series summed in reverse order.

As noted above, let $d(n)$ denote the number of positive divisors of the positive integer n. Define the "error term" $\Delta(x)$, for $x > 0$, by

$$\sum_{n\leq x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \Delta(x), \tag{2.1.20}$$

where γ denotes Euler's constant, and where the prime $'$ on the summation sign on the left side indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted. The famous *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. M.G. Voronoï [310] established a representation for $\Delta(x)$ in terms of Bessel functions with his famous formula

$$\sum_{n\leq x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{2.1.21}$$

where $x > 0$ and $I_1(z)$ is defined by (2.1.7). Since the appearance of (2.1.21) in 1904, this identity has been the starting point for most attempts at finding an upper bound for $\Delta(x)$. Readers will note a remarkable similarity between the Bessel functions in (2.1.6) and those in (2.1.21), indicating that there must be a connection between these two formulas.

From the argument that we made in (2.1.19), it is reasonable to guess that Ramanujan might have regarded the double series in (2.1.5) symmetrically, i.e., that Ramanujan really was thinking of the double sum in the form $\lim_{N\to\infty}\sum_{mn\leq N}$. Thus, as with (2.1.5), we also prove (2.1.6) with the double series being interpreted symmetrically. Our proof uses (2.1.21) and twisted, or weighted, divisor sums. Our proofs of Entry 2.1.2 under the two interpretations that we have discussed first appeared in [57].

The identities in Entries 2.1.1 and 2.1.2, with the double series interpreted as iterated double series, might give researchers new tools in approaching the circle and divisor problems, respectively. The additional parameter θ in the two primary Bessel function identities might be useful in a yet unforeseen way.

In summary, there are three ways to interpret the double series in Entries 2.1.1 and 2.1.2. Our proofs in this volume cover both entries in two of the three possible interpretations.

Analogues of the problems of estimating the error terms $P(x)$ and $\Delta(x)$ exist for many other arithmetic functions $a(n)$ generated by Dirichlet series satisfying a functional equation involving the gamma function $\Gamma(s)$. See, for example, a paper by K. Chandrasekharan and R. Narasimhan [90]. As with the cases of $r_2(n)$ and $d(n)$, representations in terms of Bessel functions for $\sum_{n\leq x} a(n)$ and more generally for $\sum_{n\leq x} a(n)(x-n)^q$, which are occasionally called Riesz sums, play a critical role. See, for example, [26, 89], and [31]. A Bessel function identity for $\sum'_{n\leq x} a(n)(x-n)^q$ is, in fact, equivalent to the functional equation involving $\Gamma(s)$ of the corresponding Dirichlet series [89]. The second author, S. Kim, and Zaharescu [59] have established a Riesz sum identity for

$$\sideset{}{'}\sum_{n\leq x}(x-n)^{\nu-1}\sum_{r|n}\sin(2\pi r\theta),$$

which in the special case $\nu = 1$ reduces to (2.1.5). One might also ask whether Ramanujan's identities in Entries 2.1.1 and 2.1.2 are isolated results, or whether they are forerunners of further theorems of this sort. To that end, the second author, S. Kim, and Zaharescu [58] have found three additional results akin to the aforementioned entries. We provide one example.

Define, for Dirichlet characters χ_1 modulo p and χ_2 modulo q,

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n}\chi_1(d)\chi_2(n/d).$$

Also, for arithmetic functions f and g, we define

$$\sideset{}{'}\sum_{nm\leq x} f(n)g(m) = \begin{cases}\sum_{nm\leq x} f(n)g(m), & \text{if } x\notin\mathbb{Z},\\ \sum_{nm\leq x} f(n)g(m) - \frac{1}{2}\sum_{nm=x} f(n)g(m), & \text{if } x\in\mathbb{Z}.\end{cases}$$

Theorem 2.1.3. *Let* $I_1(x)$ *be defined by* (2.1.7). *If* $0<\theta$, $\sigma<1$, *and* $x>0$, *then*

$$\begin{aligned}&\sum_{nm\le x}{}' \cos(2\pi n\theta)\cos(2\pi m\sigma)\\&= \frac{1}{4} + \frac{\sqrt{x}}{4}\sum_{n,m\ge 0}\left\{\frac{I_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}}\right.\\&\quad\left. + \frac{I_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}}\right\},\end{aligned} \tag{2.1.22}$$

where in the double sum on the right-hand side of (2.1.22)*, the product* mn *of the two summation indices tends to infinity.*

The remaining two theorems in [58] involve sums of products of sines and sums of products of sines and cosines, respectively. The employment of sums of $d_{\chi_1,\chi_2}(n)$ is crucial in all of the proofs.

2.2 Proof of Ramanujan's First Bessel Function Identity (Original Form)

In this section we provide a proof of Entry 2.1.1 in the form given by Ramanujan. Our proof is a more detailed exposition of the proof given by the second author, S. Kim, and Zaharescu [60]. On the other hand, these authors actually prove a more general theorem. First, they introduce a family of Dirichlet series. For $x > 0$ and $0 < \theta < 1$, let

$$G(x, \theta, s) = \sum_{m=1}^{\infty} \frac{a(x, \theta, m)}{m^s}, \tag{2.2.1}$$

where the coefficients $a(x, \theta, m)$ are given by

$$a(x, \theta, m) = \sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{n+\theta}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{n+1-\theta}}\right\}. \tag{2.2.2}$$

For $x > 0$ and $0 < \theta < 1$, by (2.1.9), the series in (2.2.1) is absolutely convergent in the half-plane $\operatorname{Re} s > \frac{5}{4}$. Second, to prove Ramanujan's claim in Entry 2.1.1, we need to establish an analytic continuation of $G(x, \theta, s)$ to a larger region. In [60], the aforementioned authors prove the following theorem.

Theorem 2.2.1. *For* $x > 0$ *and* $0 < \theta < 1$, $G(x, \theta, s)$ *has an analytic continuation to the half-plane* $\operatorname{Re} s > \frac{8}{17}$. *For* s *in this half-plane, and* $x > 0$, *the series in* (2.2.1) *converges uniformly with respect to* θ *in every compact subinterval of* $(0, 1)$. *If* x *is not an integer, these conclusions hold in the larger half-plane* $\operatorname{Re} s > \frac{1}{3}$.

Fourier analysis is then employed to recover the value of $G(x, \theta, s)$ at $s = \frac{1}{2}$, and in this way, Entry 2.1.1 is established.

2.2.1 Identifying the Source of the Poles

Fix $x > 0$, and define, for $0 < \theta < 1$,

$$g(\theta) := \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{2.2.3}$$

In order for Ramanujan's Entry 2.1.1 to be valid, the double series in (2.2.3) needs to converge, and the function $g(\theta)$ needs to be continuous on $(0,1)$. We prove this by showing that the double series converges uniformly with respect to θ in every compact subinterval of $(0,1)$. Also, in order for Entry 2.1.1 to hold, $g(\theta)$ needs to have simple poles at $\theta = 0$ and $\theta = 1$. We start by employing a heuristic argument, which allows us to identify that part of the double series that is responsible for these poles.

Setting $a = 4\pi\sqrt{x}$ and taking the terms from the right-hand side of (2.1.5) when $n = 0$, we are led to examine the series

$$T(\theta) := \sum_{m=1}^{\infty}\frac{J_1(a\sqrt{\theta m})}{\sqrt{m}}.$$

We consider the Mellin transform of $T(\theta)$, for σ sufficiently large, and make the change of variable $t^2 = a^2\theta m$ to find that

$$\begin{aligned}\int_0^{\infty} T(\theta)\theta^{s-1}d\theta &= \sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\int_0^{\infty} J_1(a\sqrt{\theta m})\theta^{s-1}d\theta\\ &= \frac{2}{a^{2s}}\sum_{m=1}^{\infty}\frac{1}{m^{s+1/2}}\int_0^{\infty} J_1(t)t^{2s-1}dt\\ &= \frac{2}{a^{2s}}\zeta(s+\tfrac{1}{2})2^{2s-1}\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{3}{2}-s)},\end{aligned} \tag{2.2.4}$$

where we used a well-known Mellin transform for Bessel functions [126, p. 707, formula 6.561, no. 14], which is valid for $-\frac{1}{2} < \sigma < \frac{3}{4}$. Applying Mellin's inversion formula in (2.2.4), for $\frac{1}{2} < c < \frac{3}{4}$, we find that

$$T(\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s+\tfrac{1}{2})\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{3}{2}-s)}\left(\frac{a^2\theta}{4}\right)^{-s}ds. \tag{2.2.5}$$

We would now like to shift the line of integration to the left of $\sigma = \frac{1}{2}$ by integrating over a rectangle with vertices $c \pm iT, b \pm iT$, where $T > 0$ and $0 < b < \frac{1}{2}$, and then letting $T \to \infty$. Thus, since the integrand has a simple pole at $s = \frac{1}{2}$ with residue

$$\left(\frac{a^2\theta}{4}\right)^{-1/2} = \frac{2}{a\sqrt{\theta}},$$

we find that

$$T(\theta) = \frac{2}{a\sqrt{\theta}} + \cdots .$$

We assume that the missing terms represented by $\cdots$ above are bounded as $\theta \to 0^+$. Returning to (2.1.5) and recalling the notation $a = 4\pi\sqrt{x}$, we find that the portion of (2.1.5) corresponding to the terms when $n = 0$ is asymptotically equal to, as $\theta \to 0^+$,

$$\frac{1}{2}\sqrt{\frac{x}{\theta}}\frac{2}{4\pi\sqrt{x\theta}} = \frac{1}{4\pi\theta}.$$

Since

$$-\frac{1}{4}\cot(\pi\theta) = -\frac{1}{4\pi\theta} + O(\theta),$$

as $\theta \to 0$, we see that the right-hand side of (2.1.5) is continuous at $\theta = 0$. A similar argument holds for $\theta = 1$.

By this heuristic argument, if we remove from the definition of $g(\theta)$ all the terms with $n = 0$, we should obtain a function that can be extended by continuity to $[0, 1]$. We prove that this is indeed the case, by showing that the sum of terms with $n \geq 1$ converges uniformly with respect to θ in $[0, 1]$. As for the terms with $n = 0$, we will show that their sum converges uniformly with respect to θ in every compact subinterval of $(0, 1)$, and that if each of these terms is multiplied by $\sin^2(\pi\theta)$, then their sum converges uniformly with respect to θ in $(0, 1)$ to a continuous function on $(0, 1)$, which tends to 0 as $\theta \to 0^+$ or $\theta \to 1^-$. If we assume that the aforementioned statements have been proved, it follows that the function $G(\theta)$ defined on $[0, 1]$ by $G(0) = 0$, $G(1) = 0$, and $G(\theta) = \sin^2(\pi\theta)g(\theta)$ is well-defined and continuous on $[0, 1]$. We return to the function $G(\theta)$ in Sect. 2.2.10. We now proceed to study the uniform convergence of the double series on the right side of Entry 2.1.1. In what follows, by "uniform convergence with respect to θ" of any series or double series below, we mean that one simultaneously has uniform convergence with respect to θ on every compact subinterval of $(0, 1)$ for the given series, and uniform convergence with respect to θ in $[0, 1]$ for the series obtained by removing the terms with $n = 0$ from the given series.

2.2.2 Large Values of n

Fix an $x > 0$, and set $a = \sqrt{4\pi x}$. With the use of (2.1.9), the problem of the uniform convergence with respect to θ of the double series on the right side of Entry 2.1.1 reduces to the study of the uniform convergence with respect to θ of the double series

$$S_1(a, \theta) := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)} - \frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right). \tag{2.2.6}$$

We first truncate the inner sum in order to further reduce the problem to one in which the summation over n is finite. Accordingly,

$$\begin{aligned}
&\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right| \\
&\quad\leq \frac{\left|\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})-\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})\right|}{(n+\theta)^{3/4}} \\
&\qquad+\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right|\left|\cos(a\sqrt{m(n+1-\theta)}-\tfrac{3\pi}{4})\right| \\
&\quad\leq \frac{|a\sqrt{m(n+\theta)}-a\sqrt{m(n+1-\theta)}|}{(n+\theta)^{3/4}}+\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right|.
\end{aligned} \tag{2.2.7}$$

For $n\geq 1$, uniformly with respect to $\theta\in[0,1]$,

$$\left|\sqrt{n+\theta}-\sqrt{n+1-\theta}\right|=O\left(\frac{1}{\sqrt{n}}\right) \tag{2.2.8}$$

and

$$\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right|=O\left(\frac{1}{n^{7/4}}\right). \tag{2.2.9}$$

Thus, by (2.2.7)–(2.2.9),

$$\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right|=O_a\left(\frac{\sqrt{m}}{n^{5/4}}\right). \tag{2.2.10}$$

(Here, and in what follows, if the constant implied by O is dependent on a parameter a, then we write O_a.) It follows that

$$\begin{aligned}
&\sum_{n\geq m^3\log^5 m}\frac{1}{m^{3/4}}\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right| \\
&\quad=O_a\left(\frac{1}{m^{1/4}}\sum_{n\geq m^3\log^5 m}\frac{1}{n^{5/4}}\right)=O_a\left(\frac{1}{m\log^{5/4}m}\right),
\end{aligned} \tag{2.2.11}$$

which shows that the sum over m at the left-hand side of (2.2.11) is convergent. Therefore the double sum $S_1(a,\theta)$ is convergent, respectively uniformly convergent, if and only if the sum

$$S_2(a,\theta) := \sum_{m=1}^{\infty} \sum_{0\leq n<m^3\log^5 m} \frac{1}{m^{3/4}} \left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}} \right) \tag{2.2.12}$$

is convergent, respectively uniformly convergent.

2.2.3 Small Values of n

Our next goal is to remove from the sum those terms in which n is much smaller than m. To this end, let us consider a general sum of the form

$$S(\alpha,\beta,\mu,H_1,H_2) := \sum_{H_1<m\leq H_2} \frac{\cos(\alpha\sqrt{m+\mu}+\beta)}{(m+\mu)^{3/4}}, \tag{2.2.13}$$

where $\alpha>0$, $\beta\in\mathbb{R}$, $\mu\in[0,1]$, and $H_1<H_2$ are large positive integers. Define

$$f(y) := \frac{\cos(\alpha\sqrt{y+\mu}+\beta)}{(y+\mu)^{3/4}}. \tag{2.2.14}$$

We fix a small real number $\delta>0$ and assume that H_1 and α satisfy the inequalities

$$c_1 \leq \alpha \leq c_2 H_1^{(1-\delta)/2}, \tag{2.2.15}$$

for some constants $c_1>0$, $c_2>0$ that depend only on a (which, in turn, depends only on x). Next, we fix a positive integer $k\geq 2$ such that

$$k\delta \geq 2. \tag{2.2.16}$$

So we may take $k=1+[2/\delta]$.

We apply the Euler–Maclaurin summation formula of order k in the form

$$S(\alpha,\beta,\mu,H_1,H_2) = \sum_{H_1<m\leq H_2} f(m) = \int_{H_1}^{H_2} \left(f(y) - \frac{(-1)^k}{k!}\psi_k(y) f^{(k)}(y) \right) dy + \sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!} \left(f^{(\ell-1)}(H_2) - f^{(\ell-1)}(H_1) \right) B_\ell, \tag{2.2.17}$$

where $f(y)$ is defined in (2.2.14), B_ℓ, $\ell\geq 0$, is the ℓth Bernoulli number, and $\psi_k(y)$ is the kth Bernoulli function, defined by

$$\psi_k(y) := -k! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (2\pi i n)^{-k} e(ny), \qquad k\geq 0, \tag{2.2.18}$$

where $e(x) = e^{2\pi i x}$. Since $k \geq 2$, the Fourier series on the right side of (2.2.18) converges absolutely.

Let us note that the integral of $f(y)$ on $[H_1, H_2]$ can be bounded via a change of variable followed by an integration by parts, namely,

$$\int_{H_1}^{H_2} f(y)dy = \int_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} \frac{2\cos(\alpha t+\beta)}{\sqrt{t}} dt$$
$$= \frac{2\sin(\alpha t+\beta)}{\alpha\sqrt{t}}\Bigg|_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} + \frac{1}{\alpha}\int_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} \frac{\sin(\alpha t+\beta)}{t^{3/2}} dt = O\left(\frac{1}{H_1^{1/4}}\right), \tag{2.2.19}$$

uniformly with respect to β and μ.

Let us also observe that for each $\ell \in \{0, 1, \ldots, k\}$, the derivative $f^{(\ell)}(y)$ can be expressed as a sum of the form

$$f^{(\ell)}(y) = \sum_{j=1}^{r_\ell} c_{\ell,j}\alpha^{a_{\ell,j}}(y+\mu)^{b_{\ell,j}} \sin(\alpha\sqrt{y+\mu}+\beta)$$
$$+ \sum_{j=1}^{r'_\ell} c'_{\ell,j}\alpha^{a'_{\ell,j}}(y+\mu)^{b'_{\ell,j}} \cos(\alpha\sqrt{y+\mu}+\beta), \tag{2.2.20}$$

where r_ℓ and r'_ℓ depend only on ℓ, the coefficients $c_{\ell,j}$ and the exponents $a_{\ell,j}$, $b_{\ell,j}$ depend only on ℓ and j, and similarly, $c'_{\ell,j}$ and the exponents $a'_{\ell,j}$, $b'_{\ell,j}$ depend only on ℓ and j. Consider the collection of all pairs $(a_{\ell,j}, b_{\ell,j})$, $1 \leq j \leq r_\ell$, and $(a'_{\ell,j}, b'_{\ell,j})$, $1 \leq j \leq r'_\ell$, and denote this collection by $\mathcal{C}_\ell$. Differentiating (2.2.3) with respect to y, and taking into account the possible cancellation of terms, we conclude that $\mathcal{C}_{\ell+1}$ is a subset of the set of all pairs of the form $(a, b-1)$ and $(a+1, b-\frac{1}{2})$, with $(a,b) \in \mathcal{C}_\ell$. Taking also into account that $\mathcal{C}_0$ consists of the single pair $(0, -\frac{3}{4})$ and using induction on ℓ, we see that each pair (a,b) in $\mathcal{C}_\ell$ satisfies $0 \leq a \leq \ell$ and $-\ell - \frac{3}{4} \leq b \leq -\frac{1}{2}\ell - \frac{3}{4}$. As a consequence, we derive that for each ℓ and for each $y \in [H_1, H_2]$,

$$f^{(\ell)}(y) = O_\ell\left(\frac{1}{y^{3/4}} \cdot \left(\frac{\alpha}{\sqrt{y}}\right)^\ell\right), \tag{2.2.21}$$

uniformly with respect to β and μ. Therefore, recalling (2.2.15), we find that

$$\sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!}\left(f^{(\ell-1)}(H_2) - f^{(\ell-1)}(H_1)\right) B_\ell = O_k\left(\frac{1}{H_1^{3/4}}\right), \tag{2.2.22}$$

uniformly with respect to β and μ. Also,

$$\left|\int_{H_1}^{H_2} \frac{(-1)^k}{k!}\psi_k(y) f^{(k)}(y)dy\right| = O_k\left(\int_{H_1}^{H_2} |f^{(k)}(y)|dy\right)$$
$$= O_k\left(\int_{H_1}^{H_2} \frac{1}{y^{3/4}}\left(\frac{\alpha}{\sqrt{y}}\right)^k dy\right) = O_k\left(\frac{\alpha^k}{H_1^{\frac{1}{2}k-\frac{1}{4}}}\right) = O_{a,k}\left(\frac{1}{H_1^{\frac{1}{2}k\delta-\frac{1}{4}}}\right). \tag{2.2.23}$$

Thus, by (2.2.16), (2.2.17), (2.2.19), (2.2.22), and (2.2.23), we find that, subject to (2.2.15) holding,

$$\sum_{H_1<m\le H_2} \frac{\cos(\alpha\sqrt{m+\mu}+\beta)}{(m+\mu)^{3/4}} = O_{a,k}\left(\frac{1}{\alpha H_1^{1/4}}\right). \tag{2.2.24}$$

We now consider the sum

$$S_3(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{0\le n<m^{1-\delta}} \frac{1}{m^{3/4}} \left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}} \right). \tag{2.2.25}$$

For each $M \ge 1$, we denote by $S_{3,M}(a,\theta,\delta)$ the corresponding restricted sum in $S_3(a,\theta,\delta)$, where the summation over m is restricted to $1 \le m \le M$. We intend to show that the sum $S_3(a,\theta,\delta)$ is convergent, and in order to do this, we apply Cauchy's criterion. Fix $\epsilon > 0$. We need to show that there exists an M_ϵ such that for every $M_1, M_2 > M_\epsilon$,

$$|S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)| < \epsilon. \tag{2.2.26}$$

Let $M_1 < M_2$ be large, and interchange the order of summation to rewrite $S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)$ in the form

$$S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta) = \sum_{0\le n\le M_2^{1-\delta}} \left(\sum_{\max\{n^{1/(1-\delta)},M_1\}<m\le M_2} \times \frac{1}{(n+\theta)^{3/4}} \frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{m^{3/4}} - \frac{1}{(n+1-\theta)^{3/4}} \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{m^{3/4}} \right). \tag{2.2.27}$$

Using (2.2.24) with $\beta = -3\pi/4$, $\mu = 0$, $H_1 = \max\{n^{1/(1-\delta)}, M_1\}$, $H_2 = M_2$, and $\alpha = a\sqrt{n+\theta}, a\sqrt{n+1-\theta}$, respectively, and noting that (2.2.15) holds, we conclude from (2.2.24) that

$$|S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)| = O_{a,\delta}\left(\sum_{0\le n\le M_2^{1-\delta}} \left\{ \frac{1}{(n+\theta)^{3/4}} \cdot \frac{1}{\sqrt{n+\theta}\,(\max\{n^{1-\delta},M_1\})^{1/4}} + \frac{1}{(n+1-\theta)^{3/4}} \cdot \frac{1}{\sqrt{n+1-\theta}\,\left(\max\{n^{1/(1-\delta)},M_1\}\right)^{1/4}} \right\}\right)$$

$$= O_{a,\delta}\left(\sum_{0\le n\le M_2^{1-\delta}} \frac{1}{n^{5/4}\max\{n^{1/(4(1-\delta))}, M_1^{1/4}\}}\right)$$

$$= O_{a,\delta}\left(\sum_{0\le n\le M_1^{1-\delta}} \frac{1}{n^{5/4}M_1^{1/4}}\right) + O_{a,\delta}\left(\sum_{M_1^{1-\delta}<n\le M_2^{1-\delta}} \frac{1}{n^{5/4+1/(4(1-\delta))}}\right)$$

$$= O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right) + O_{a,\delta}\left(\frac{1}{(M_1^{1-\delta})^{1/4+1/(4(1-\delta))}}\right)$$

$$= O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right) + O_{a,\delta}\left(\frac{1}{M_1^{(1-\delta)/4+1/4}}\right) = O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right). \tag{2.2.28}$$

The foregoing analysis implies (2.2.26) for M_1 sufficiently large, and proves the convergence of $S_3(a,\theta,\delta)$. Therefore the convergence of $S_1(a,\theta,\delta)$ reduces to the convergence, respectively uniform convergence, of

$$S_4(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta}\le n<m^3\log^5 m} \frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right). \tag{2.2.29}$$

2.2.4 Further Reductions

The remaining series under consideration, $S_4(a,\theta,\delta)$, does not contain any terms with $n=0$. Therefore, in what follows, uniform convergence means uniform convergence with respect to θ in $[0,1]$. Next, we write

$$S_4(a,\theta,\delta) = S_5(a,\theta,\delta) + S_6(a,\theta,\delta), \tag{2.2.30}$$

where $S_5(a,\theta,\delta)$ denotes the sum of those terms in $S_4(a,\theta,\delta)$ for which $n>m^{1+\delta}$, and $S_6(a,\theta,\delta)$ is the sum of terms with $n\le m^{1+\delta}$. The examination of $S_5(a,\theta,\delta)$ is like that for $S_3(a,\theta,\delta)$. In this case, we take the sum over n as the inner sum, and apply (2.2.24) with $\beta=-3\pi/4$, $\alpha=a\sqrt{m}$, and $\mu=\theta,1-\theta$, respectively. We accordingly find that the sum $S_5(a,\theta,\delta)$ converges uniformly with respect to θ. It follows that the convergence of $S_1(a,\theta,\delta)$ reduces to that of $S_6(a,\theta,\delta)$.

Let us consider now the sum

$$S_7(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta}\le n\le m^{1+\delta}} \frac{\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)\sin\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)}{m^{3/4}n^{3/4}}. \tag{2.2.31}$$

We claim that $S_6(a,\theta,\delta)$ is uniformly convergent if and only if $S_7(a,\theta,\delta)$ is. Indeed,

$$\frac{1}{(n+\theta)^{3/4}} = \frac{1}{n^{3/4}} + O\left(\frac{1}{n^{7/4}}\right), \qquad \frac{1}{(n+1-\theta)^{3/4}} = \frac{1}{n^{3/4}} + O\left(\frac{1}{n^{7/4}}\right), \tag{2.2.32}$$

and it is easily seen that the error terms in (2.2.32) are small enough so that the denominators $(n+\theta)^{3/4}$ and $(n+1-\theta)^{3/4}$ in $S_6(a,\theta,\delta)$ can both be replaced by $n^{3/4}$ without influencing the uniform convergence of the sum. Also,

$$\begin{aligned} \cos&\left(a\sqrt{m(n+\theta)} - \frac{3\pi}{4}\right) - \cos\left(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4}\right) \\ &= 2\sin\left(\frac{a\sqrt{m}\left(\sqrt{n+1-\theta} - \sqrt{n+\theta}\right)}{2}\right) \\ &\quad \times \sin\left(\frac{a\sqrt{m}\left(\sqrt{n+1-\theta} + \sqrt{n+\theta}\right)}{2} - \frac{3\pi}{4}\right). \end{aligned} \tag{2.2.33}$$

Here

$$\sqrt{n+1-\theta} - \sqrt{n+\theta} = \frac{1-2\theta}{2\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \tag{2.2.34}$$

and

$$\frac{\sqrt{n+1-\theta} + \sqrt{n+\theta}}{2} = \sqrt{n+\frac{1}{2}} + O\left(\frac{1}{n^{3/2}}\right). \tag{2.2.35}$$

By (2.2.33)–(2.2.35),

$$\begin{aligned} \cos&\left(a\sqrt{m(n+\theta)} - \frac{3\pi}{4}\right) - \cos\left(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4}\right) \\ &= 2\sin\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right) + O\left(\frac{\sqrt{m}}{n^{3/2}}\right). \end{aligned} \tag{2.2.36}$$

Again, it is easily checked that the error term in (2.2.36) is small enough so that the left side of (2.2.36) may be replaced by the main term from the right side of (2.2.36) in the modified version of $S_6(a,\theta,\delta)$ above without influencing the uniform convergence of the series. This proves our claim, and it remains to show the uniform convergence of $S_7(a,\theta,\delta)$.

We replace $S_7(a,\theta,\delta)$ by another series that has the same terms, but the double summation is performed over a union of rectangles. To be precise, for each positive integer r, we consider those m satisfying the inequalities $2^r \le m < 2^{r+1}$, and for each such m we replace the range of summation for n, which in $S_7(a,\theta,\delta)$ is $m^{1-\delta} \le n \le m^{1+\delta}$, with the somewhat larger range $2^{r(1-\delta)} \le n \le 2^{(r+1)(1+\delta)}$. This does not influence the uniform convergence of the series, because the extra terms added by this procedure are contained in the sums $S_3(a,\theta,\delta)$ and $S_5(a,\theta,\delta)$, which we have previously examined. More specifically, the extra terms arise from the ranges $2^{r(1-\delta)} \le n < m^{1-\delta}$ and $m^{1+\delta} < n \le 2^{(r+1)(1+\delta)}$. In both these ranges, either n is significantly smaller than m ($n < m^{1-\delta}$), or n is significantly larger than m ($n > m^{1+\delta}$), and so an appropriate use of (2.2.24) can be made in both cases. In conclusion, $S_7(a,\theta,\delta)$ is uniformly convergent if and only if the same is true for the sum

$$S_8(a,\theta,\delta) := \sum_{r=1}^{\infty} \sum_{2^r \le m < 2^{r+1}} \sum_{2^{r(1-\delta)} \le n \le 2^{(r+1)(1+\delta)}} \times \frac{\sin\left(b\sqrt{\frac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}, \tag{2.2.37}$$

where, henceforth, we define, for simplicity,

$$b = \frac{a(1-2\theta)}{4} = \pi\sqrt{x}(1-2\theta). \tag{2.2.38}$$

2.2.5 Refining the Range of Summation

In order to prove that $S_8(a,\theta,\delta)$ is uniformly convergent with respect to θ in $[0,1]$, we need to show that the right side of (2.2.37) converges uniformly with respect to b in $[-\pi\sqrt{x}, \pi\sqrt{x}]$. To do this, we use Cauchy's criterion. Fix $\epsilon > 0$ and denote, as usual, for any $M > 1$, the partial sum in (2.2.37) corresponding to $1 \le m \le M$ by $S_{8,M}(a,\theta,\delta)$. Let $M_1 < M_2$ be large, and set $r_1 = [\log_2 M_1]$ and $r_2 = [\log_2 M_2]$. Then $S_{8,M_2}(a,\theta,\delta) - S_{8,M_1}(a,\theta,\delta)$ can be written as a sum over integral pairs (m,n) in the union of $r_2 - r_1 + 1$ rectangles, which we denote by $R_0, R_1, \ldots, R_{r_2-r_1}$, as follows. We let $R_0 = \left(M_1, 2^{r_1+1}\right) \times \left[2^{r_1(1-\delta)}, 2^{(r_1+1)(1+\delta)}\right]$, $R_j = \left[2^{r_1+j}, 2^{r_1+j+1}\right) \times \left[2^{(r_1+j)(1-\delta)}, 2^{(r_1+j+1)(1+\delta)}\right]$ for $1 \le j \le r_2 - r_1 - 1$, and $R_{r_2-r_1} = [2^{r_2}, M_2] \times \left[2^{r_2(1-\delta)}, 2^{r_2(1+\delta)}\right]$. Then

$$S_{8,M_2}(a,\theta,\delta) - S_{8,M_1}(a,\theta,\delta) = \sum_{j=0}^{r_2-r_1} \sum_{(m,n)\in R_j} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}. \tag{2.2.39}$$

We now proceed to obtain bounds for the inner sum on the right side of (2.2.39) for each individual R_j. Fix such an R_j, and, to make a choice, assume that $1 \le j \le r_2 - r_1 - 1$. The cases $j = 0$ and $j = r_2 - r_1$ can be examined in a similar fashion. Also, for simplicity, we set $T = 2^{r_1+j}$. Then the corresponding inner sum on the right side of (2.2.39), which depends on a, b, δ, and T, and which we denote by $\sum_{a,b,\delta,T}$, or simply by $\sum$, has the form

$$\sum_{a,b,\delta,T} = \sum_{T\leq m<2T} \sum_{T^{1-\delta}\leq n\leq (2T)^{1+\delta}} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}. \tag{2.2.40}$$

At this point, we fix a number λ, with $0<\lambda<\frac{1}{2}$, whose precise value will be given later, and set $L=[T^{\lambda}]$. Then we subdivide the rectangle $[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$ into squares of size $L\times L$. An explanation as to why we break the range of summation into such small squares of size $[T^{\lambda}]\times[T^{\lambda}]$, with $\lambda<\frac{1}{2}$, is in order. This choice may seem surprising, because for almost all exponential sums, the best one can hope to achieve is a square-root-type cancellation. And in our case, square-root cancellation over a square of size $[T^{\lambda}]\times[T^{\lambda}]$ means a savings over the trivial bound by a factor of T^{λ}. But this is not enough in our case, even if we achieve a square-root cancellation for each individual square of size $L\times L$, because the trivial bound for the entire sum $\sum_{a,b,\delta,T}$, even ignoring the small but strictly positive δ, is of order $O(T^{1/2})$. Thus we need cancellation in $\sum_{a,b,\delta,T}$ by a factor larger than $T^{1/2}$, and so a cancellation by a factor of T^{λ} with $\lambda<\frac{1}{2}$ will not suffice.

Our approach below, which proceeds via subdividing the range of summation into small squares of size $[T^{\lambda}]\times[T^{\lambda}]$, with $\lambda<\frac{1}{2}$, is based on two fundamental ideas. The first one is that on such small squares, the functions $(m,n)\mapsto m^{-3/4}n^{-3/4}$ and $(m,n)\mapsto b\sqrt{m/n}$ are almost constant, and the function $a\sqrt{m(n+\frac{1}{2})}$ is almost linear. This gives us a chance to approximate locally the corresponding sums on the right side of (2.2.40) by geometric series, for which we have better than square-root cancellation. The second idea is to approximate the function

$$(m,n)\mapsto \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}$$

by a short sum in which each term is a product of a function of m and a function of n. This, in turn, reduces the problem of bounding the right side of (2.2.40) to the problem of bounding certain sums that are products of a sum over m and a sum over n. This gives us the opportunity to combine the savings achieved due to cancellation in the sum over m with the savings achieved in the sum over n.

To proceed, we consider the set of integral points (m,n) in $[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$ for which both m and n are divisible by L. We also consider all the squares of size $L\times L$ with vertices in the aforementioned set. These squares almost cover the rectangle above. We first examine the portion of the rectangle left uncovered, and bound its contribution on the right side of (2.2.40).

Let

$$T_1 := \left[\frac{T}{L}\right]+1, \quad T_2 := \left[\frac{2T}{L}\right]-1, \quad T_3 := \left[\frac{T^{1-\delta}}{L}\right]+1, \quad T_4 := \left[\frac{(2T)^{1+\delta}}{L}\right]-1. \tag{2.2.41}$$

For each $m_1 \in \{T_1, T_1+1, \ldots, T_2\}$ and $n_1 \in \{T_3, T_3+1, \ldots, T_4\}$, we consider the $L \times L$ square whose southwest corner has coordinates (Lm_1, Ln_1), and denote by $\sum_{m_1,n_1}$ its contribution on the right-hand side of (2.2.40). To be precise, we define

$$\sum_{m_1,n_1} := \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \times \frac{\sin\left(b\sqrt{\dfrac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\dfrac{1}{2}\right)} - \dfrac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}. \tag{2.2.42}$$

Then we approximate the right side of (2.2.40) by the sum $\sum_{m_1,n_1}$ with (m_1, n_1) running over the pairs of integral points in the rectangle $[T_1, T_2] \times [T_3, T_4]$. The error made in this approximation is bounded as follows. Note that each integral point (m,n) in $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$ that does not belong to any of the $L \times L$ squares of the form $[Lm_1, L(m_1+1)) \times [Ln_1, L(n_1+1))$, with $T_1 \le m_1 \le T_2$, $T_3 \le n_1 \le T_4$, is at distance at most L from one of the four sides of the rectangle $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$. Therefore,

$$\begin{aligned}
&\left| \sum_{a,b,\delta,T} - \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \sum_{m_1,n_1} \right| \\
&= O\left(\sum_{\substack{|m-T| \le L \\ \text{or} \\ |m-2T| \le L}} \sum_{T^{1-\delta} \le n \le (2T)^{1+\delta}} \frac{1}{m^{3/4} n^{3/4}} \right) \\
&\quad + O\left(\sum_{\substack{|n-T^{1-\delta}| \le L \\ \text{or} \\ |n-(2T)^{1+\delta}| \le L}} \sum_{T \le m \le 2T} \frac{1}{m^{3/4} n^{3/4}} \right) \\
&= O\left(\frac{L}{T^{3/4}} \cdot T^{(1+\delta)/4} \right) + O\left(\frac{L}{T^{3(1-\delta)/4}} \cdot T^{1/4} \right) \\
&= O\left(\frac{L}{T^{\frac{1}{2}-\frac{3}{4}\delta}} \right) \\
&= O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}} \right).
\end{aligned} \tag{2.2.43}$$

In our approach, we first fix λ, and then we fix δ depending on λ. In particular, δ is chosen small enough so that $\frac{1}{2}-\lambda-\frac{3}{4}\delta>0$, which ensures that the far right side of (2.2.43) is negligible.

Next, we proceed to bound each sum $\sum_{m_1,n_1}$. Fix $m_1\in\{T_1,T_1+1,\dots,T_2\}$ and $n_1\in\{T_3,T_3+1,\dots,T_4\}$. For each m and n, with $Lm_1\le m<L(m_1+1)$ and $Ln_1\le n<L(n_1+1)$, we find that, with several uses of (2.2.41) below,

$$\frac{1}{m^{3/4}}=\frac{1}{L^{3/4}m_1^{3/4}\left(1+O(1/m_1)\right)}=\frac{1}{L^{3/4}m_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda}}\right)\right), \quad (2.2.44)$$

$$\frac{1}{n^{3/4}}=\frac{1}{L^{3/4}n_1^{3/4}\left(1+O(1/n_1)\right)}=\frac{1}{L^{3/4}n_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \quad (2.2.45)$$

$$\sqrt{\frac{m}{n}}=\frac{\sqrt{Lm_1}\cdot(1+O(1/m_1))}{\sqrt{Ln_1}\cdot(1+O(1/n_1))}=\sqrt{\frac{m_1}{n_1}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \quad (2.2.46)$$

and, noting the definition of b given in (2.2.38), we further see that

$$\begin{aligned}\sin\left(b\sqrt{\frac{m}{n}}\right)&=\sin\left(b\sqrt{\frac{m_1}{n_1}}+O\left(\frac{|b|\sqrt{m_1}}{\sqrt{n_1}\,T^{1-\lambda-\delta}}\right)\right)\\&=\sin\left(b\sqrt{\frac{m_1}{n_1}}\right)+O_x\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right), \quad (2.2.47)\end{aligned}$$

uniformly with respect to θ in $[0,1]$. Hence, by (2.2.42) and (2.2.44)–(2.2.47),

$$\begin{aligned}\sum_{m_1,n_1}&=\sum_{Lm_1\le m<L(m_1+1)}\ \sum_{Ln_1\le n<L(n_1+1)}\frac{1}{L^{3/2}m_1^{3/4}n_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right)\\&\times\left(\sin\left(b\sqrt{\frac{m_1}{n_1}}\right)+O_x\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right)\right)\cdot\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)\\&=\frac{\sin\left(b\sqrt{m_1/n_1}\right)}{L^{3/2}m_1^{3/4}n_1^{3/4}}\sum_{Lm_1\le m<L(m_1+1)}\ \sum_{Ln_1\le n<L(n_1+1)}\\&\times\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)+O_x\left(\frac{\sqrt{L}}{m_1^{3/4}n_1^{3/4}T^{1-\lambda-\frac{3}{2}\delta}}\right). \quad (2.2.48)\end{aligned}$$

Here,

$$m_1^{3/4}n_1^{3/4}\ge T_1^{3/4}T_3^{3/4}>\left(\frac{T}{L}\right)^{3/4}\left(\frac{T^{1-\delta}}{L}\right)^{3/4}\sim T^{\frac{3}{2}-\frac{3}{2}\lambda-\frac{3}{4}\delta}. \quad (2.2.49)$$

By (2.2.48) and (2.2.49),

$$\left|\sum_{m_1,n_1}\right| \ll \frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \left| \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \right. \\ \left. \times \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)\right| + O_x\left(\frac{1}{T^{\frac{5}{2}-3\lambda-\frac{9}{4}\delta}}\right). \quad (2.2.50)$$

2.2.6 Short Exponential Sums

Consider now the exponential sum

$$E_{m_1,n_1} := \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} e\left(2\sqrt{xm(n+\tfrac{1}{2})}\right), \quad (2.2.51)$$

where, as customary, $e(t) := e^{2\pi i t}$. Observe that

$$\sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right) \\ = \mathrm{Im}\left(e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right). \quad (2.2.52)$$

Since

$$\left|\mathrm{Im}\left(e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right)\right| \le \left|e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right| = |E_{m_1,n_1}|,$$

by (2.2.50), we see that

$$\left|\sum_{m_1,n_1}\right| = O\left(\frac{|E_{m_1,n_1}|}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\right) + O\left(\frac{1}{T^{\frac{5}{2}-3\lambda-\frac{9}{4}\delta}}\right). \quad (2.2.53)$$

Adding the estimates (2.2.53) for all relevant values of m_1 and n_1 and using the bound

$$T_2T_4 = O\left(T^{2-2\lambda+\delta}\right),$$

we see that

$$\sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \left|\sum_{m_1,n_1}\right| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} |E_{m_1,n_1}|\right) \\ + O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \quad (2.2.54)$$

From (2.2.43) and (2.2.54), we deduce that

$$\left|\sum_{a,b,\delta,T}\right| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{T_1\le m_1\le T_2} \sum_{T_3\le n_1\le T_4} |E_{m_1,n_1}|\right) + O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \tag{2.2.55}$$

For fixed $\lambda < \frac{1}{2}$ and δ small enough so that $\frac{1}{2}-\lambda-\frac{13}{4}\delta > 0$, the second error term on the right-hand side of (2.2.55) is negligible. In order to estimate the first error term on the right side of (2.2.55), fix m_1 and n_1. We write each m and n with $Lm_1 \le m < L(m_1+1)$ and $Ln_1 \le n < L(n_1+1)$ in the forms

$$m = Lm_1 + m_2, \qquad n = Ln_1 + n_2, \qquad m_2, n_2 \in \{0, 1, \ldots, L-1\}. \tag{2.2.56}$$

Then,

$$\begin{aligned}\sqrt{m} &= \sqrt{Lm_1}\left(1+\frac{m_2}{Lm_1}\right)^{1/2}\\ &= \sqrt{Lm_1}\left(1+\frac{m_2}{2Lm_1}-\frac{m_2^2}{8L^2m_1^2}+O\left(\frac{m_2^3}{L^3m_1^3}\right)\right)\\ &= \sqrt{Lm_1}\left(1+\frac{m_2}{2Lm_1}-\frac{m_2^2}{8L^2m_1^2}+O\left(\frac{1}{T^{3-3\lambda}}\right)\right),\end{aligned} \tag{2.2.57}$$

$$\begin{aligned}\sqrt{n+\tfrac{1}{2}} &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{Ln_1}\right)^{1/2}\\ &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{2Ln_1}-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{(n_2+\frac{1}{2})^3}{L^3n_1^3}\right)\right)\\ &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{2Ln_1}-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{1}{T^{3-3\lambda-3\delta}}\right)\right).\end{aligned} \tag{2.2.58}$$

Also, by (2.2.41) and (2.2.56),

$$\frac{m_2}{2Lm_1}\cdot\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2} = O\left(\frac{L}{T}\cdot\frac{L^2}{T^{2-2\delta}}\right) = O\left(\frac{1}{T^{3-3\lambda-2\delta}}\right), \tag{2.2.59}$$

$$\frac{m_2^2}{8L^2m_1^2}\cdot\frac{(n_2+\frac{1}{2})}{2Ln_1} = O\left(\frac{L^2}{T^2}\cdot\frac{L}{T^{1-\delta}}\right) = O\left(\frac{1}{T^{3-3\lambda-\delta}}\right), \tag{2.2.60}$$

$$\frac{m_2^2}{8L^2m_1^2}\cdot\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2} = O\left(\frac{L^2}{T^2}\cdot\frac{L^2}{T^{2-2\delta}}\right) = O\left(\frac{1}{T^{4-4\lambda-2\delta}}\right). \tag{2.2.61}$$

By (2.2.57)–(2.2.61),

$$\begin{aligned}\sqrt{m(n+\tfrac{1}{2})} = L\sqrt{m_1n_1}\Bigg(1&+\frac{m_2}{2Lm_1}+\frac{n_2+\frac{1}{2}}{2Ln_1}+\frac{m_2(n_2+\frac{1}{2})}{4L^2m_1n_1}-\frac{m_2^2}{8L^2m_1^2}\\ &-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{1}{T^{3-3\lambda-3\delta}}\right)\Bigg).\end{aligned} \tag{2.2.62}$$

Next, again with the use of (2.2.41) and (2.2.56),

$$L\sqrt{m_1n_1}\cdot\frac{1}{T^{3-3\lambda-3\delta}}=O\left(\frac{T^{1+\delta/2}}{T^{3-3\lambda-3\delta}}\right)=O\left(\frac{1}{T^{2-3\lambda-7\delta/2}}\right), \tag{2.2.63}$$

$$L\sqrt{m_1n_1}\cdot\frac{\frac{1}{2}m_2}{4L^2m_1n_1}=O\left(\frac{T^{\lambda}}{L\sqrt{m_1n_1}}\right)=O\left(\frac{1}{T^{1-\lambda-\delta/2}}\right), \tag{2.2.64}$$

$$L\sqrt{m_1n_1}\cdot\frac{n_2+\frac{1}{4}}{8L^2n_1^2}=O\left(\frac{\sqrt{m_1n_1}}{n_1^2}\right)=O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right). \tag{2.2.65}$$

By (2.2.62)–(2.2.65), we see that

$$\sqrt{m(n+\tfrac{1}{2})}=L\sqrt{m_1n_1}\left(1+\frac{1}{4Ln_1}\right)+\frac{1}{2}\sqrt{\frac{n_1}{m_1}}\cdot m_2+\frac{1}{2}\sqrt{\frac{m_1}{n_1}}\cdot n_2$$
$$-\frac{\sqrt{m_1n_1}}{8L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2+O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right)+O\left(\frac{1}{T^{2-3\lambda-7\delta/2}}\right). \tag{2.2.66}$$

Note that for

$$3\delta<1-2\lambda,$$

which we may assume in what follows, $T^{2-3\lambda-7\delta/2}>T^{1-\lambda-\delta/2}$. Therefore, by (2.2.66), we find that

$$e\left(-2L\sqrt{xm_1n_1}\left(1+\frac{1}{4Ln_1}\right)\right)e\left(2\sqrt{xm(n+\tfrac{1}{2})}\right)$$
$$=e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right)e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right)e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right)$$
$$+O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right). \tag{2.2.67}$$

Summing up the relations (2.2.67) over m_2 and n_2 in their appropriate ranges, taking absolute values on both sides, and recalling (2.2.51), we find that

$$|E_{m_1,n_1}|=\left|e\left(-2L\sqrt{xm_1n_1}\left(1+\frac{1}{4Ln_1}\right)\right)\cdot E_{m_1,n_1}\right|$$
$$=\left|\sum_{0\le m_2<L}\sum_{0\le n_2<L}e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right)e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right)\right.$$
$$\left.\times\, e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right)\right|$$
$$+O\left(T^{3\lambda-1+3\delta/2}\right). \tag{2.2.68}$$

We now use the Taylor expansion for

$$e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right).$$

Observe that

$$\begin{aligned}\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2 &\leq \frac{\sqrt{xm_1n_1}}{4L}\max\left\{\frac{m_2^2}{m_1^2},\frac{n_2^2}{n_1^2}\right\}\\ &\leq \frac{\sqrt{xm_1n_1}}{4L}\max\left\{\frac{L^2}{m_1^2},\frac{L^2}{n_1^2}\right\}\\ &= \frac{L\sqrt{x}}{4}\max\left\{\frac{n_1^{1/2}}{m_1^{3/2}},\frac{m_1^{1/2}}{n_1^{3/2}}\right\}\\ &= O_x\left(T^{\lambda}\cdot\max\left\{\frac{T^{(1+\delta-\lambda)/2}}{T^{3(1-\lambda)/2}},\frac{T^{(1-\lambda)/2}}{T^{3(1-\delta-\lambda)/2}}\right\}\right)\\ &= O_x\left(\frac{1}{T^{1-2\lambda-3\delta/2}}\right). \qquad (2.2.69)\end{aligned}$$

In what follows we fix a positive integer r, depending on λ only, such that $(r+1)(\frac{1}{2}-\lambda)\geq 1$. For example, we may take

$$r=\left[\frac{1}{\frac{1}{2}-\lambda}\right]. \qquad (2.2.70)$$

We also assume that δ is small enough so that

$$3\delta < 1-2\lambda.$$

Then $1-2\lambda-\frac{3}{2}\delta > \frac{1}{2}-\lambda$, and so by (2.2.69),

$$\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2 = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda}}\right). \qquad (2.2.71)$$

We may then truncate the Taylor series expansion mentioned above as

$$\begin{aligned}&e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right)\\ &=\sum_{j=0}^{r}\frac{(-1)^j(xm_1n_1)^{j/2}}{4^jL^jj!}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^{2j}+O_{x,\lambda,\delta}\left(\frac{1}{T^{(r+1)(\frac{1}{2}-\lambda)}}\right)\\ &=\sum_{j=0}^{r}\sum_{\ell=0}^{2j}\frac{(-1)^jx^{j/2}}{4^jj!}\binom{2j}{\ell}\frac{m_1^{\frac{1}{2}j-\ell}n_1^{\ell-\frac{3}{2}j}}{L^j}m_2^{\ell}n_2^{2j-\ell}+O_{x,\lambda,\delta}\left(\frac{1}{T}\right), \qquad (2.2.72)\end{aligned}$$

by (2.2.71). Inserting (2.2.72) in (2.2.68), and noticing that the error term on the right side of (2.2.72) is small enough so that when inserted on the right side of (2.2.68) it can be subsumed in the existing error term from (2.2.68), we deduce that

$$|E_{m_1,n_1}| = \left|\sum_{j=0}^{r}\sum_{\ell=0}^{2j} A_{j,\ell}(m_1,n_1)V_{j,\ell}(m_1,n_1)\right| + O_{x,\lambda,\delta}\left(T^{3\lambda+\frac{3}{2}\delta-1}\right), \quad (2.2.73)$$

where we have defined

$$A_{j,\ell}(m_1,n_1) := \frac{(-1)^j x^{j/2}}{4^j j!}\binom{2j}{\ell}\frac{m_1^{\frac{1}{2}j-\ell}\, n_1^{\ell-\frac{3}{2}j}}{L^j} \quad (2.2.74)$$

and

$$V_{j,\ell}(m_1,n_1) := \sum_{0\le m_2<L}\sum_{0\le n_2<L} e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right) e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right) m_2^{\ell} n_2^{2j-\ell}. \quad (2.2.75)$$

In order to bound the coefficients $A_{j,\ell}(m_1,n_1)$, we distinguish two cases: $\ell \ge 3j/2$ and $\ell < 3j/2$. If $\ell \ge 3j/2$, in order to produce an upper bound for the right side of (2.2.74), we need an upper bound for n_1, which is $T^{1-\lambda+\delta}$. When $\ell < 3j/2$, we need a lower bound for n_1, which is $T^{1-\lambda-\delta}$. For m_1, both upper and lower bounds have the same size, $T^{1-\lambda}$. Combining the two cases, we find that

$$\begin{aligned} |A_{j,\ell}(m_1,n_1)| &= O_{x,\lambda,\delta}\left(\frac{T^{(1-\lambda)(\frac{1}{2}j-\ell)}\cdot T^{(1\pm\delta-\lambda)(\ell-\frac{3}{2}j)}}{T^{\lambda j}}\right) \\ &= O_{x,\lambda,\delta}\left(\frac{1}{T^{j-\frac{3}{2}\delta j}}\right), \end{aligned} \quad (2.2.76)$$

uniformly for $\ell \in \{0,1,\ldots,2j\}$.

The exponential sum on the right-hand side of (2.2.75), as hinted earlier, can be written as the product of two exponential sums, each in one variable,

$$V_{j,\ell}(m_1,n_1) = \left(\sum_{0\le m_2<L} e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right) m_2^{\ell}\right)\left(\sum_{0\le n_2<L} e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right) n_2^{2j-\ell}\right). \quad (2.2.77)$$

In the case $j = \ell = 0$, the exponential sums above are geometric series, which can be accurately estimated. For any real number α, any integer M, and any positive integer H, we recall the well-known uniform upper bound

$$\left|\sum_{n=M+1}^{M+H} e(\alpha n)\right| = O\left(\min\left\{H, \frac{1}{\|\alpha\|}\right\}\right), \quad (2.2.78)$$

where $\|\alpha\|$ denotes the distance from α to the nearest integer. Using (2.2.78) in (2.2.77), we find that, for $j = \ell = 0$,

$$|V_{0,0}(m_1, n_1)| = O\left(\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} \cdot \min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right). \tag{2.2.79}$$

For general j and ℓ, a familiar argument based on (2.2.78) in combination with summation by parts for each of the two exponential sums on the right-hand side of (2.2.77) gives

$$|V_{j,\ell}(m_1, n_1)| = O_{j,\ell}\left(L^{2j}\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} \cdot \min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right). \tag{2.2.80}$$

Using (2.2.76) and (2.2.80) for all $0 \le j \le r$, $0 \le \ell \le 2j$ and defining r by (2.2.70), we find from (2.2.73) that

$$\begin{aligned}|E_{m_1,n_1}| = O_{x,\lambda,\delta}\Bigg(&\sum_{0\le j\le[1/(\frac{1}{2}-\lambda)]}\sum_{\ell=0}^{2j}\frac{L^{2j}}{T^{j-\frac{3}{2}\delta j}} \cdot \min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\\ &\times\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\Bigg) + O\left(T^{3\lambda+\frac{3}{2}\delta-1}\right).\end{aligned} \tag{2.2.81}$$

Here $L^2/T^{1-\frac{3}{2}\delta} < 1$ for $\delta < \frac{2}{3}(1-2\lambda)$, which we assume in the sequel, and so the maximum value of $L^{2j}/T^{j-\frac{3}{2}\delta j}$ is attained at $j = 0$. Thus, by (2.2.81),

$$\begin{aligned}|E_{m_1,n_1}| = O_{x,\lambda,\delta}\Bigg(&\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\\ &\times\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\Bigg) + O\left(T^{3\lambda+\frac{3}{2}\delta-1}\right).\end{aligned} \tag{2.2.82}$$

Next, we employ (2.2.82) on the right side of (2.2.55). In doing so, note that the error term on the right side of (2.2.82) produces an error term on the right side of (2.2.55) that is bounded by

$$O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \cdot T_2 \cdot T_4 \cdot T^{3\lambda+\frac{3}{2}\delta-1}\right) = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).$$

This is smaller than the existing error term on the right side of (2.2.55), and we deduce that

$$\left|\sum_{a,b,\delta,T}\right| = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{T_1\le m_1\le T_2}\sum_{T_3\le n_1\le T_4}\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right.$$
$$\left.\times\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \quad (2.2.83)$$

2.2.7 Uniform Convergence When x Is Not an Integer

Our next idea is based on the observation that if for some m_1 and n_1, both $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are simultaneously small, thus producing a large term on the right side of (2.2.83), then each of $\sqrt{xn_1/m_1}$ and $\sqrt{xm_1/n_1}$ is close to an integer, and hence their product is correspondingly close to an integer. But their product equals x, which is fixed throughout the proof, so this event cannot happen unless x is an integer. Fix an x that is not an integer. Then $\|x\| = \min\{|x-y| : y \in \mathbb{Z}\} > 0$. For each $m_1 \in \{T_1, \dots, T_2\}$ and $n_1 \in \{T_3, \dots, T_4\}$, let d_1 and d_2 be integers, depending on m_1 and n_1, such that

$$\left\|\sqrt{\frac{xn_1}{m_1}}\right\| = \left|d_1 - \sqrt{\frac{xn_1}{m_1}}\right| \quad (2.2.84)$$

and

$$\left\|\sqrt{\frac{xm_1}{n_1}}\right\| = \left|d_2 - \sqrt{\frac{xm_1}{n_1}}\right|. \quad (2.2.85)$$

Using (2.2.84) and (2.2.85) and the fact that d_1d_2 is an integer, we find that

$$\|x\| \le |x - d_1d_2| = \left|\sqrt{\frac{xn_1}{m_1}}\sqrt{\frac{xm_1}{n_1}} - d_1d_2\right| \le \left|\left(\sqrt{\frac{xn_1}{m_1}} - d_1\right)\sqrt{\frac{xm_1}{n_1}}\right|$$
$$+\left|d_1\left(\sqrt{\frac{xm_1}{n_1}} - d_2\right)\right| = \sqrt{\frac{xm_1}{n_1}}\left\|\sqrt{\frac{xn_1}{m_1}}\right\| + d_1\left\|\sqrt{\frac{xm_1}{n_1}}\right\|. \quad (2.2.86)$$

Here,

$$\sqrt{\frac{xm_1}{n_1}} = O_{x,\delta}(T^{\delta}) \quad (2.2.87)$$

and

$$d_1 \le \sqrt{\frac{xn_1}{m_1}} + \frac{1}{2} = O_{x,\delta}(T^{\delta}). \quad (2.2.88)$$

Thus, by (2.2.86)–(2.2.88),

$$\|x\| = O_{x,\delta}\left(T^{\delta}\max\left\{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|, \left\|\sqrt{\frac{xm_1}{n_1}}\right\|\right\}\right). \quad (2.2.89)$$

It follows from (2.2.89) that, uniformly for $m_1 \in \{T_1, \ldots, T_2\}$ and $n_1 \in \{T_3, \ldots, T_4\}$,

$$\min\left\{\frac{1}{\|\sqrt{xn_1/m_1}\|}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\} = O_{x,\delta}(T^{\delta}). \tag{2.2.90}$$

By (2.2.90), it follows that

$$\begin{aligned}
&\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} \cdot \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\} \\
&\quad = O_{x,\delta}\left(T^{\delta}\left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} + \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right)\right).
\end{aligned} \tag{2.2.91}$$

Inserting (2.2.91) into the right side of (2.2.83), we deduce that

$$\begin{aligned}
\left|\sum_{a,b,\delta,T}\right| &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}} \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right. \\
&\quad \left. + \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).
\end{aligned} \tag{2.2.92}$$

The summation in the first of the two error terms on the right side of (2.2.92) yields two double sums. For the first, we keep the order of summation as in (2.2.92) and focus on the inner sum

$$F(x, \delta, \lambda, T, m_1) := \sum_{T_3 \le n_1 \le T_4} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}, \tag{2.2.93}$$

while for the second, we interchange the order of summation, so that the inner sum becomes

$$G(x, \delta, \lambda, T, n_1) := \sum_{T_1 \le m_1 \le T_2} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}. \tag{2.2.94}$$

We proceed to derive an upper bound for $F(x, \delta, \lambda, T, m_1)$. Each term in the sum on the right side of (2.2.93) lies in $[2, T^{\lambda}]$. We subdivide this interval into dyadic intervals $[2, 4), [4, 8), \ldots,\ [2^s, T^{\lambda}]$, where $s = [\lambda \log_2 T]$. For each $j = 1, 2, \ldots, s$, set

$$B_{j,m_1} := \left\{n_1 \in \{T_3, \ldots, T_4\} : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]\right\}. \tag{2.2.95}$$

Then

$$F(x,\delta,\lambda,T,m_1) \le \sum_{j=1}^{s} 2^{j+1}\#\{B_{j,m_1}\} + T^{\lambda}\#\left\{T_3 \le n_1 \le T_4 : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \le \frac{1}{T^{\lambda}}\right\}. \quad (2.2.96)$$

Now fix $j \in \{1,2,\dots,s\}$. We need an accurate upper bound for $\#\{B_{j,m_1}\}$. For each $n_1 \in B_{j,m_1}$, we let, as before, d_1 denote the closest integer to $\sqrt{xn_1/m_1}$. Then using (2.2.41), as we often have done and will continue to do, we see that, for T sufficiently large,

$$0 \le d_1 \le \sqrt{\frac{xT_4}{T_1}} + \frac{1}{2} \le \sqrt{2^{1+\delta}xT^{\delta}} + \frac{1}{2} \le \sqrt{3xT^{\delta}}. \quad (2.2.97)$$

By (2.2.95) and (2.2.97), it follows that

$$\begin{aligned} B_{j,m_1} \subseteq & \bigcup_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \{T_3 \le n_1 \le T_4 : \\ & \sqrt{\frac{xn_1}{m_1}} \in \left[d_1 - \frac{1}{2^j}, d_1 - \frac{1}{2^{j+1}}\right] \cup \left[d_1 + \frac{1}{2^{j+1}}, d_1 + \frac{1}{2^j}\right]\Big\} \\ \subseteq & \bigcup_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \left\{T_3 \le n_1 \le T_4 : n_1 \in \left[\frac{m_1}{x}\left(d_1 - \frac{1}{2^j}\right)^2, \frac{m_1}{x}\left(d_1 + \frac{1}{2^j}\right)^2\right]\right\}. \end{aligned} \quad (2.2.98)$$

For each interval I of real numbers,

$$\#\{\mathbb{Z} \cap I\} \le 1 + \mathrm{length}(I). \quad (2.2.99)$$

From (2.2.98) and (2.2.99), we find that

$$\begin{aligned} \#\{B_{j,m_1}\} &\le \sum_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \left(1 + \frac{m_1}{x}\left(d_1 + \frac{1}{2^j}\right)^2 - \frac{m_1}{x}\left(d_1 - \frac{1}{2^j}\right)^2\right) \\ &= 1 + \left[\sqrt{3xT^{\delta}}\right] + \frac{m_1}{x}\sum_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \frac{4d_1}{2^j} \\ &= O_x\left(T^{\delta/2}\right) + O_x\left(\frac{m_1}{2^j} \cdot T^{\delta}\right). \end{aligned} \quad (2.2.100)$$

Similarly,

$$\#\left\{T_3 \le n_1 \le T_4 : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \le \frac{1}{T^{\lambda}}\right\} = O_x\left(T^{\delta/2}\right) + O_x\left(\frac{m_1}{T^{\lambda}} \cdot T^{\delta}\right). \quad (2.2.101)$$

Employing (2.2.100) and (2.2.101) on the right-hand side of (2.2.96), we deduce that

$$\begin{aligned}F(x,\delta,\lambda,T,m_1) &= O_x\left(\sum_{j=1}^{s} 2^{j+1}\left(T^{\delta/2}+\frac{m_1}{2^j}T^{\delta}\right)\right)\\&\quad + O_x\left(T^{\lambda}\left(T^{\delta/2}+\frac{m_1}{T^{\lambda}}T^{\delta}\right)\right)\\&= O_x(2^sT^{\delta/2}+sm_1T^{\delta})+O_x(T^{\lambda+\delta/2}+m_1T^{\delta})\\&= O_{x,\lambda,\delta}(T^{\lambda+\delta/2}+m_1T^{\delta}\log T), \qquad (2.2.102)\end{aligned}$$

where we have recalled the definition $s=[\lambda\log_2 T]$. Reversing the roles of m_1 and n_1, using the same argument as above, appealing to (2.2.41), and invoking the inequalities

$$0\le d_2\le\sqrt{\frac{xm_1}{n_1}}+\frac{1}{2}\le\sqrt{\frac{2xT_2}{T_3}}+\frac{1}{2}=O_{x,\delta}(T^{\delta/2}) \qquad (2.2.103)$$

in place of (2.2.97), we also deduce that

$$G(x,\delta,\lambda,T,n_1)=O_{x,\lambda,\delta}(T^{\lambda+\delta/2}+n_1T^{\delta}\log T). \qquad (2.2.104)$$

Combining (2.2.92)–(2.2.94), (2.2.102), and (2.2.104), we find that

$$\begin{aligned}\left|\sum_{a,b,\delta,T}\right| &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\sum_{T_1\le m_1\le T_2}\left(T^{\lambda+\delta/2}+m_1T^{\delta}\log T\right)\right)\\&\quad+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\sum_{T_3\le n_1\le T_4}\left(T^{\lambda+\delta/2}+n_1T^{\delta}\log T\right)\right)\\&\quad+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right)\\&= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\left(T^{1+\delta/2}+T^{2-2\lambda+\delta}\log T\right)\right)\\&\quad+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\left(T^{1+3\delta/2}+T^{2-2\lambda+3\delta}\log T\right)\right)\\&\quad+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right)\\&= O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \qquad (2.2.105)\end{aligned}$$

So far, the only condition we have put on λ is that $\lambda\in(0,\frac{1}{2})$. We now see that in order for the argument above to work, we also need $\lambda>\frac{1}{4}$. Then the first error term on the far right side of (2.2.105) will be small enough. In order to balance the exponents not involving δ in the two error terms on the far right side of (2.2.105), we now choose $\lambda = \frac{1}{3}$. Then by (2.2.105) and the fact that $\log T$ is smaller than $T^{\delta/4}$ for fixed $\delta > 0$ and T sufficiently large,

$$\left| \sum_{a,b,\delta,T} \right| = O_{x,\delta}\left(\frac{1}{T^{\frac{1}{6}-5\delta}} \right). \tag{2.2.106}$$

Let us recall that $\sum_{a,b,\delta,T}$ is one of the inner sums on the right-hand side of (2.2.40), with, from the discourse prior to (2.2.40), $T = 2^{r_1+j}$. Using (2.2.106) for each of these sums and recalling the definition $r_1 = [\log_2 M_1]$, we find from (2.2.39) that

$$\begin{aligned} |S_{8,M_2}(a,\theta,\delta) - S_{8,M_1}(a,\theta,\delta)| &= O_{x,\delta}\left(\sum_{j=0}^{r_2-r_1} \frac{1}{2^{(r_1+j)(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{2^{r_1(\frac{1}{6}-5\delta)}} \sum_{j=0}^{\infty} \frac{1}{2^{j(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{2^{r_1(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{M_1^{(\frac{1}{6}-5\delta)}} \right), \end{aligned} \tag{2.2.107}$$

uniformly with respect to θ in $[0,1]$. This completes the proof that the sum $S_8(a,\theta,\delta)$ converges uniformly with respect to θ in $[0,1]$, which in turn implies a corresponding statement for the initial double sum $S_1(a,\theta)$.

2.2.8 The Case That x Is an Integer

We now proceed to examine the case that x is an integer. In the case above, in which x is not an integer, the relations (2.2.106) and consequently (2.2.107) were stronger than needed, in the sense that a weaker savings, where the exponent $\frac{1}{6}$ is replaced by any smaller strictly positive constant, would have sufficed. The fact that we had some room to spare in the proof above naturally leads us to expect that exactly the same argument as above would cover as well, at least partially, the case that x is an integer. With this in mind, we subdivide the sum $\sum_{a,b,\delta,T}$ into two sums, one for which the argument above applies, to be examined first, and the second, to be examined later.

We begin by fixing a positive integer x. Next, we fix an arbitrary small real number $\eta > 0$. With η fixed, we then choose a real number $\lambda < \frac{1}{2}$, depending on η. The exact dependence of λ on η will be clarified later, with the crux of the matter being that λ is chosen such that $\frac{1}{2}-\lambda$ is much smaller than η. With η and λ fixed, we then choose $\delta > 0$, depending on η and λ. The dependence of δ on η and λ will be made explicit later, with the goal being that δ will be chosen to be much smaller than $\frac{1}{2} - \lambda$. Once η, λ, and δ are fixed, we start by following the same reduction procedure from the foregoing beginning of the proof, which reduces the convergence, respectively uniform convergence, of $S_1(a, \theta)$ to that of $S_8(a, \theta, \delta)$. In order to investigate the convergence of $S_8(a, \theta, \delta)$, we again employ Cauchy's criterion, and arrive at (2.2.40). We need to show that the right side of (2.2.40) is in absolute value less than ϵ, for an arbitrary fixed $\epsilon > 0$. We again bound each of the inner sums on the right-hand side of (2.2.40) separately. As before, we fix j, with $1 \leq j \leq r_2 - r_1 - 1$, set $T = 2^{r_1+j}$, and consider the sum $\sum_{a,b,\delta,T}$, defined in (2.2.42). At this point, we divide the sum $\sum_{a,b,\delta,T}$ into two parts, depending on η, as follows. Consider in $\mathbb{R}^2$ the rectangle

$$D(\delta, T) := [T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}].$$

For each divisor d of x, draw the ray from the origin with slope d^2/x. Around this ray, consider the thin trapezoidal region, say $V(x, d, \eta, \delta, T)$, that consists of all the points in $D(\delta, T)$ for which the slope of the line from the origin through the point lies in the interval

$$\left[\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}\right]. \tag{2.2.108}$$

Set

$$U_1(a, b, \delta, T, \eta) := \sum_{(m,n)\in D(\delta,T)\setminus\cup_{d|x} V(x,d,\eta,\delta,T)} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}} \tag{2.2.109}$$

and

$$U_2(a, b, \delta, T, \eta) := \sum_{d|x} \sum_{(m,n)\in V(x,d,\eta,\delta,T)} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}. \tag{2.2.110}$$

Thus,

$$\sum_{a,b,\delta,T} = U_1(a, b, \delta, T, \eta) + U_2(a, b, \delta, T, \eta). \tag{2.2.111}$$

Before proceeding further, we observe that although the trapezoids $V(x,d,\eta,\delta,T)$ are very thin, we cannot afford to trivially estimate $U_2(a,b,\delta,T,\eta)$, as we did in (2.2.45). Indeed, $V(x,d,\eta,\delta,T)$ is a trapezoid with (horizontal) height T lying inside an angle of measure roughly $1/T^{\frac{1}{2}-\eta}$, and so the two bases have size of the order of magnitude of $T^{\frac{1}{2}+\eta}$. Therefore the area of $V(x,d,\eta,\delta,T)$ is of order $T^{\frac{3}{2}+\eta}$. The number of integral points (m,n) in $V(x,d,\eta,\delta,T)$ is asymptotic to this area, since the perimeter of the trapezoid is of smaller order, $O(T)$. On the other hand, the denominator $m^{3/4}n^{3/4}$ on the right side of (2.2.110) is of precise order of magnitude $T^{3/2}$. To see this, note that n/m lies between $1/x$ and x. For other points in the trapezoid $V(x,d,\eta,\delta,T)$, for T sufficiently large, n/m lies between $1/2x$ and $2x$, say. Since $T<m<2T$, this implies that $T/2x<n<4xT$. Therefore, if we estimate the sum on the right side of (2.2.110) trivially, we obtain

$$|U_2(a,b,\delta,T,\eta)| = O_{x,\eta,\delta}(T^{\eta}), \tag{2.2.112}$$

which is not sufficient for our purposes. This discussion also shows that any cancellation on the right side of (2.2.110) allowing us to save a factor of T^{c_0}, for some constant $c_0>0$ independent of η, would suffice (by taking η smaller than c_0).

Taking into account the shape of these trapezoids, we see that it does not appear appropriate to consider subdividing them into small squares as before. Instead, it is more natural to try to achieve cancellation on large exponential sums taken along parallel lines of corresponding slope d^2/x, which is what we will do later.

We first bound $U_1(a,b,\delta,T,\eta)$. Subdivide $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$ into squares of size $L\times L$, where, as before, $L=[T^{\lambda}]$. Let T_1,T_2,T_3, and T_4 be as defined in (2.2.41). For each $m_1\in\{T_1,\dots,T_2\}$ and $n_1\in\{T_3,\dots,T_4\}$, we define $\sum_{m_1,n_1}$ by (2.2.42). We consider all those squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$ for which the lower left corner does not belong to any of the trapezoids $V(x,d,\eta,\delta,T)$. Since the slope of the ray from the origin to this lower left corner equals n_1/m_1, the condition above can be stated as

$$\frac{n_1}{m_1}\notin \cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]. \tag{2.2.113}$$

Note that all the integral points (m,n) in $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$ that do not belong to the union of squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$, $m_1\in\{T_1,\dots,T_2\}$ and $n_1\in\{T_3,\dots,T_4\}$, and that satisfy (2.2.113) are at a distance $O(L)$ from the boundary of $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$. We bound the contribution of these points (m,n) on the right side of (2.2.109) as follows. The contribution of those points (m,n) that are at a distance $O(L)$ from the four edges of the rectangle $D(\delta,T)$ was estimated in (2.2.43), and it was found to be

$$O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}}\right).$$

The remaining points, namely, those (m,n) lying inside the rectangle $D(\delta,T)$ that are at a distance $O(L)$ from the union over $d \mid x$ of the rays from the origin of slopes

$$\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}} \quad \text{and} \quad \frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}$$

can be bounded in a similar manner. One then finds that their contribution to the right side of (2.2.109) is

$$O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda}}\right).$$

Combining all these bounds, we deduce that

$$\left| U_1(a,b,\delta,T,\eta) - \sum_{\substack{T_1\le m_1\le T_2\\ T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]}} \sum_{m_1,n_1} \right| = O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}}\right). \tag{2.2.114}$$

Next, we apply (2.2.53) to each $\sum_{m_1,n_1}$ in (2.2.114), and obtain a relation analogous to (2.2.55), namely,

$$|U_1(a,b,\delta,T,\eta)| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{\substack{T_1\le m_1\le T_2\\ T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]}} |E_{m_1,n_1}|\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right), \tag{2.2.115}$$

where E_{m_1,n_1} is defined in (2.2.51). The exponential sums E_{m_1,n_1} were bounded in (2.2.82). Employing those bounds on the right-hand side of (2.2.115), we derive a relation analogous to (2.2.83), namely,

$$|U_1(a,b,\delta,T,\eta)| = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{\substack{T_1 \le m_1 \le T_2 \\ T_3 \le n_1 \le T_4 \\ \frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]}} \min\left\{T^\lambda, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} \cdot \min\left\{T^\lambda, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right)$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \tag{2.2.116}$$

Unlike the previous case, in which x was not an integer and $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ cannot be simultaneously small, in the present case in which x is an integer, $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ can be small simultaneously. This can happen only if n_1/m_1 is close to a number of the form d^2/x with $d|x$. Conversely, if n_1/m_1 is close to d^2/x for some divisor d of x, then automatically m_1/n_1 is close to d'^2/x, where $dd' = x$, and $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are simultaneously small. The extra condition on n_1/m_1 in the summation on the right side of (2.2.116) assures us that $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ cannot be simultaneously small. This does not prevent the possibility that one of $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ is much smaller than the other, of course. But in that case, the other is larger than $1/T^\delta$, by (2.2.41), and so the term corresponding to the pair (m_1, n_1) on the right side of (2.2.116) is harmless, as we have seen before.

With this in mind, we proceed as follows. Consider the sets of integral points (m_1, n_1) defined by

$$\mathcal{B}_1(x,\eta,\lambda,\delta,T) := \Bigg\{(m_1,n_1) : T_1 \le m_1 \le T_2, T_3 \le n_1 \le T_4,$$
$$\frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right], \max\left\{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|, \left\|\sqrt{\frac{xm_1}{n_1}}\right\|\right\} > \frac{1}{T^\delta}\Bigg\} \tag{2.2.117}$$

and

$$\mathcal{B}_2(x,\eta,\lambda,\delta,T) := \Bigg\{(m_1,n_1) : T_1 \le m_1 \le T_2, T_3 \le n_1 \le T_4,$$
$$\frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right], \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \le \frac{1}{T^\delta}, \left\|\sqrt{\frac{xm_1}{n_1}}\right\| \le \frac{1}{T^\delta}\Bigg\}. \tag{2.2.118}$$

The last condition in the definition of $\mathcal{B}_1(x,\eta,\lambda,\delta,T)$ is equivalent to

$$\min\left\{\frac{1}{\|\sqrt{xn_1/m_1}\|},\frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}<T^{\delta},\tag{2.2.119}$$

which is analogous to (2.2.90). Therefore the contribution of $\mathcal{B}_1(x,\eta,\lambda,\delta,T)$ on the right side of (2.2.116) can be estimated as in the previous case when x was not an integer. In the present case, we arrive at (2.2.91) and proceed similarly as in the proof that previously led to (2.2.105), but now there remains the estimate of the summation over (m_1,n_1) in $\mathcal{B}_2(x,\eta,\lambda,\delta,T)$. Accordingly, up to this point, we obtain the bounds

$$\begin{aligned}
&|U_1(a,b,\delta,T,\eta)|\\
&=O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\min\left\{T^{\lambda},\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right.\\
&\quad\left.\times\min\left\{T^{\lambda},\frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right)+O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).
\end{aligned}\tag{2.2.120}$$

Next, let us observe that for each $(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)$, if we denote by d_1 and d_2 the closest integers to $\sqrt{xn_1/m_1}$ and $\sqrt{xm_1/n_1}$, respectively, then

$$\begin{aligned}
|d_1d_2-x|&=\left|\left(d_1-\sqrt{\frac{xn_1}{m_1}}\right)d_2+\sqrt{\frac{xn_1}{m_1}}\left(d_2-\sqrt{\frac{xm_1}{n_1}}\right)\right|\\
&=\left\|\sqrt{\frac{xn_1}{m_1}}\right\||d_2+\sqrt{\frac{xn_1}{m_1}}\left\|\sqrt{\frac{xm_1}{n_1}}\right\|.
\end{aligned}\tag{2.2.121}$$

Here, by (2.2.41),

$$\sqrt{\frac{xn_1}{m_1}}=O_x(T^{\delta/2})\qquad\text{and}\qquad d_2=\sqrt{\frac{xm_1}{n_1}}+O(1)=O_x(T^{\delta/2}),$$

while

$$\left\|\sqrt{\frac{xn_1}{m_1}}\right\|\leq\frac{1}{T^{\delta}}\qquad\text{and}\qquad\left\|\sqrt{\frac{xm_1}{n_1}}\right\|\leq\frac{1}{T^{\delta}},$$

by (2.2.118). On using the foregoing estimates in (2.2.121), we find that

$$|d_1d_2-x|=O_x\left(\frac{1}{T^{\delta/2}}\right),\tag{2.2.122}$$

and since d_1, d_2, and x are integers, (2.2.122) implies that $d_1d_2=x$. Let us further observe that for $(m_1,n_1)\in\mathbb{B}_2(x,\eta,\lambda,\delta,T)$, the quantities $1/\|\sqrt{xn_1/m_1}\|$

and $1/\|\sqrt{xm_1/n_1}\|$, which are both larger than T^δ by (2.2.118), have the same order of magnitude. Indeed,

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{\left|d_1-\sqrt{\frac{xn_1}{m_1}}\right|}{\left|d_2-\sqrt{\frac{xm_1}{n_1}}\right|}=\frac{\left|d_1^2-\frac{xn_1}{m_1}\right|\left(d_2+\sqrt{\frac{xm_1}{n_1}}\right)}{\left|d_2^2-\frac{xm_1}{n_1}\right|\left(d_1+\sqrt{\frac{xn_1}{m_1}}\right)}. \tag{2.2.123}$$

Here, by (2.2.41),

$$d_2+\sqrt{\frac{xm_1}{n_1}}=2d_2+O\left(\frac{1}{T^\delta}\right)=2d_2\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.124}$$

$$d_1+\sqrt{\frac{xn_1}{m_1}}=2d_1\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.125}$$

$$\left|d_1^2-\frac{xn_1}{m_1}\right|=\frac{1}{m_1}\left|d_1^2m_1-d_1d_2n_1\right|=\frac{d_1}{m_1}|d_1m_1-d_2n_1|, \tag{2.2.126}$$

and

$$\left|d_2^2-\frac{xm_1}{n_1}\right|=\frac{d_2}{n_1}|d_2n_1-d_1m_1|. \tag{2.2.127}$$

By (2.2.123)–(2.2.127), we see that unless $d_2n_1=d_1m_1$,

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{n_1}{m_1}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right). \tag{2.2.128}$$

But

$$\frac{n_1}{m_1}=\frac{\sqrt{\frac{xn_1}{m_1}}}{\sqrt{\frac{xm_1}{n_1}}}=\frac{d_1+O_x\left(\frac{1}{T^\delta}\right)}{d_2+O_x\left(\frac{1}{T^\delta}\right)}=\frac{d_1}{d_2}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right). \tag{2.2.129}$$

By (2.2.128) and (2.2.129), it follows that

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{d_1}{d_2}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.130}$$

unless $d_2n_1=d_1m_1$, in which case both quantities $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are equal to zero. In both cases, we can conclude that

$$\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\} = O_x\left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right). \tag{2.2.131}$$

Inserting (2.2.131) into the right-hand side of (2.2.120), we find that

$$\begin{aligned}
&|U_1(a,b,\delta,T,\eta)| \\
&= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right)^2\right) \\
&\quad + O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).
\end{aligned} \tag{2.2.132}$$

We proceed to estimate the sum in the first error term on the right-hand side of (2.2.132). Recall that for any $(m_1,n_1)\in\mathbb{B}_2(x,\eta,\lambda,\delta,T)$, on the one hand, $\|\sqrt{xn_1/m_1}\|\le 1/T^{\delta}$, and on the other hand,

$$\frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right],$$

and so, in particular,

$$\left|\frac{n_1}{m_1}-\frac{d_1^2}{x}\right|\ge\frac{1}{T^{\frac{1}{2}-\eta}}, \tag{2.2.133}$$

where as before, d_1 is the closest integer to $\sqrt{xn_1/m_1}$. By (2.2.133) and (2.2.125),

$$\begin{aligned}
\left\|\sqrt{\frac{xn_1}{m_1}}\right\| &= \left|d_1-\sqrt{\frac{xn_1}{m_1}}\right| = \frac{x\left|\frac{d_1^2}{x}-\frac{n_1}{m_1}\right|}{d_1+\sqrt{\frac{xn_1}{m_1}}} \ge \frac{x}{T^{\frac{1}{2}-\eta}\left(d_1+\sqrt{\frac{xn_1}{m_1}}\right)} \\
&= \frac{x}{2d_1T^{\frac{1}{2}-\eta}}\left(1+O_x\left(\frac{1}{T^{\delta}}\right)\right) > \frac{1}{4T^{\frac{1}{2}-\eta}},
\end{aligned} \tag{2.2.134}$$

for sufficiently large T.

We next subdivide the interval

$$\left[\frac{1}{4T^{\frac{1}{2}-\eta}}, \frac{1}{T^{\delta}}\right]$$

into dyadic intervals of the form

$$\left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right],$$

and, for each j, bound the contribution to the first O-term on the right side of (2.2.132) of those pairs (m_1, n_1) for which

$$\left\| \sqrt{\frac{xn_1}{m_1}} \right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right].$$

Recall that $\frac{1}{2} - \lambda$ is smaller than η, and so $1/T^\lambda < 1/(4T^{\frac{1}{2}-\eta})$. Hence,

$$\min\left\{ T^\lambda, \frac{1}{\|\sqrt{xn_1/m_1}\|} \right\} = \frac{1}{\|\sqrt{xn_1/m_1}\|} \tag{2.2.135}$$

for all $(m_1, n_1) \in \mathbb{B}_2(x, \eta, \lambda, \delta, T)$. In conclusion, if we set $s_1 := [\delta \log_2 T]$ and $s_2 := 2 + \left[\left(\frac{1}{2} - \eta\right) \log_2 T\right]$, then, with the use of (2.2.99) below,

$$\begin{aligned}
&\sum_{(m_1,n_1)\in\mathbb{B}_2(x,\eta,\lambda,\delta,T)} \left(\min\left\{ T^\lambda, \frac{1}{||\sqrt{xn_1/m_1}||} \right\} \right)^2 \\
&\quad \leq \sum_{j=s_1}^{s_2} 2^{2j+2} \# \left\{ (m_1, n_1) \in \mathbb{B}_2(x, \eta, \lambda, \delta, T) : \left\| \sqrt{\frac{xn_1}{m_1}} \right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right] \right\} \\
&\quad \leq \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \leq m_1 \leq T_2} \# \left\{ n_1 : \left| \sqrt{\frac{xn_1}{m_1}} - d \right| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right] \right\} \\
&\quad \leq \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \leq m_1 \leq T_2} \# \left\{ \mathbb{Z} \cap \left[\frac{m_1}{x} \left(d - \frac{1}{2^j} \right)^2, \frac{m_1}{x} \left(d + \frac{1}{2^j} \right)^2 \right] \right\} \\
&\quad \leq \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \leq m_1 \leq T_2} \left(1 + \frac{dm_1}{x2^{j-2}} \right) \\
&\quad = O_x \left(\sum_{j=s_1}^{s_2} 2^{2j} T_2 \right) + O_x \left(\sum_{j=s_1}^{s_2} 2^j \sum_{T_1 \leq m_1 \leq T_2} m_1 \right) \\
&\quad = O_{x,\eta,\delta,\lambda} \left(2^{2s_2} T^{1-\lambda} \right) + O_{x,\eta,\delta,\lambda} \left(2^{s_2} T^{2-2\lambda} \right) \\
&\quad = O_{x,\eta,\delta,\lambda}(T^{2-2\eta-\lambda}) + O_{x,\eta,\delta,\lambda}(T^{\frac{5}{2}-\eta-2\lambda}).
\end{aligned} \tag{2.2.136}$$

Combining (2.2.136) and (2.2.132), we finally deduce that

$$\begin{aligned}
|U_1(a, b, \delta, T, \eta)| &= O_{x,\lambda,\delta,\eta} \left(\frac{1}{T^{\lambda+2\eta-\frac{1}{2}-\frac{3}{4}\delta}} \right) + O_{x,\lambda,\delta,\eta} \left(\frac{1}{T^{2\lambda+\eta-1-\frac{3}{4}\delta}} \right) \\
&\quad + O_{x,\lambda,\delta} \left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}} \right) + O_{x,\lambda,\delta} \left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}} \right).
\end{aligned} \tag{2.2.137}$$

We now see that for any fixed $\eta > 0$, we can make all the O-terms on the right side of (2.2.137) sufficiently small by choosing λ close to $\frac{1}{2}$ and then choosing $\delta > 0$ small enough. To be precise, we fix a small $\eta > 0$, and then let $\lambda = \frac{1}{2} - \frac{1}{3}\eta$. Thus, (2.2.137) takes the shape

$$\begin{aligned}|U_1(a,b,\delta,T,\eta)| &= O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{5}{3}\eta-\frac{3}{4}\delta}}\right) + O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{3}{4}\delta}}\right)\\ &\quad + O_{x,\delta,\eta}\left(\frac{\log T}{T^{\frac{1}{2}-\frac{2}{3}\eta-\frac{19}{4}\delta}}\right) + O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{13}{4}\delta}}\right)\\ &= O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{13}{4}\delta}}\right). \end{aligned} \tag{2.2.138}$$

We now let $\delta = \eta/39$, and so from (2.2.137) we can now deduce that

$$|U_1(a,b,T,\eta)| = O_{x,\eta}\left(\frac{1}{T^{\eta/4}}\right), \tag{2.2.139}$$

where, for simplicity, we deleted the symbol δ on the left-hand side of (2.2.139), because δ is a function of η.

There remains the problem of obtaining a suitable bound for the sum $U_2(a,b,\delta,T,\eta)$. As above, we delete δ from the notations $V(x,d,\eta,\delta,T)$ and $U_2(a,b,\delta,T,\eta)$, which we now proceed to estimate.

2.2.9 Estimating $U_2(a,b,T,\eta)$

In order to bound $U_2(a,b,T,\eta)$, we estimate, for each divisor d of x, the inner sum on the right side of (2.2.110). For each $(m,n) \in V(x,d,\eta,T)$, by (2.2.108),

$$\left|\frac{n}{m} - \frac{d^2}{x}\right| \leq \frac{1}{T^{\frac{1}{2}-\eta}}. \tag{2.2.140}$$

By (2.2.140),

$$\sin\left(b\sqrt{\frac{m}{n}}\right) = \sin\left(\frac{b\sqrt{x}}{d}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right) \tag{2.2.141}$$

and

$$\frac{1}{m^{3/4}n^{3/4}} = \frac{x^{3/4}}{d^{3/2}m^{3/2}}\left(1 + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right)\right). \tag{2.2.142}$$

Hence, by (2.2.110),

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{\sin\left(\frac{b\sqrt{x}}{d}\right)}{d^{3/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(a\sqrt{m(n+\frac{1}{2})} - \frac{3\pi}{4}\right)}{m^{3/2}}$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}} \sum_{d|x} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{3/2}}\right). \tag{2.2.143}$$

Recall from the reasoning leading to (2.2.112) that the number of integral pairs (m,n) in each $V(x,d,\eta,T)$ is of the order of $T^{\frac{3}{2}+\eta}$. Thus, using this estimate in the O-term above and recalling that $T \le m < 2T$, we find that (2.2.143) reduces to

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{\sin\left(\frac{b\sqrt{x}}{d}\right)}{d^{3/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(a\sqrt{m(n+\frac{1}{2})} - \frac{3\pi}{4}\right)}{m^{3/2}}$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right). \tag{2.2.144}$$

From the inequalities $T \le m < 2T$ combined with (2.2.140), it follows that

$$\left| n - \frac{d^2 m}{x} \right| \le 2T^{\frac{1}{2}+\eta} \tag{2.2.145}$$

and

$$\begin{aligned}
\sqrt{m(n+\tfrac{1}{2})} &= \sqrt{m\left(\frac{d^2m}{x} + \frac{1}{2} + n - \frac{d^2m}{x}\right)} \\
&= \frac{dm}{\sqrt{x}}\left(1 + \frac{x\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)}{d^2m}\right)^{1/2} \\
&= \frac{dm}{\sqrt{x}}\left(1 + \frac{x\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)}{2d^2m} - \frac{x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{8d^4m^2}\right. \\
&\qquad \left. + O_x\left(\frac{\left|\frac{1}{2}+n-\frac{d^2m}{x}\right|^3}{m^3}\right)\right) \\
&= \frac{dm}{\sqrt{x}} + \frac{\sqrt{x}}{4d} + \frac{\sqrt{x}\,n}{2d} - \frac{dm}{2\sqrt{x}} - \frac{x^{3/2}\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{8d^3m} + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\right).
\end{aligned} \tag{2.2.146}$$

Recall that $a = 4\pi\sqrt{x}$. Therefore,

$$\begin{aligned}
&\sin\left(a\sqrt{m(n+\tfrac{1}{2})} - \frac{3\pi}{4}\right)\\
&\quad = \sin\left(2\pi dm + \frac{\pi x}{d} + \frac{2\pi xn}{d} - \frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} - \frac{3\pi}{4}\right)\\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\right).
\end{aligned} \tag{2.2.147}$$

Here, $2\pi dm + 2\pi xn/d$ is an integral multiple of 2π, and $\pi x/d$ is an integral multiple of π, which is a multiple of 2π if and only if x/d is even. It follows from (2.2.147) and (2.2.144) that

$$\begin{aligned}
&U_2(a,b,T,\eta)\\
&\quad = x^{3/4}\sum_{d|x}\frac{1}{d^{3/2}}\sin\left(\frac{b\sqrt{x}}{d}\right)\\
&\qquad \times \sum_{(m,n)\in V(x,d,\eta,T)}\frac{(-1)^{x/d}}{m^{3/2}}\sin\left(-\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} - \frac{3\pi}{4}\right)\\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\sum_{d|x}\sum_{(m,n)\in V(x,d,\eta,T)}\frac{1}{m^{3/2}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right)\\
&\quad = x^{3/4}\sum_{d|x}\frac{(-1)^{x/d+1}}{d^{3/2}}\sin\left(\frac{b\sqrt{x}}{d}\right)\\
&\qquad \times \sum_{(m,n)\in V(x,d,\eta,T)}\frac{1}{m^{3/2}}\sin\left(\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} + \frac{3\pi}{4}\right)\\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right).
\end{aligned} \tag{2.2.148}$$

Furthermore, by (2.2.145) and the inequalities $T \le m < 2T$,

$$\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} = \frac{\pi(xn-d^2m)^2}{2d^3m} + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right), \tag{2.2.149}$$

which, when inserted in (2.2.148), gives

$$U_2(a,b,T,\eta) = x^{3/4}\sum_{d|x}\frac{(-1)^{x/d+1}}{d^{3/2}}\sin\left(\frac{b\sqrt{x}}{d}\right)$$
$$\times \sum_{(m,n)\in V(x,d,\eta,T)}\frac{1}{m^{3/2}}\sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m}+\frac{3\pi}{4}\right)$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.150}$$

Next, for each divisor d of x, consider the function $H_d(u,v)$ of two real variables defined on $[T,2T)\times[T^{1-\delta},(2T)^{1+\delta}]$ by

$$H_d(u,v) := \frac{1}{u^{3/2}}\sin\left(\frac{\pi(xv-d^2u)^2}{2d^3u}+\frac{3\pi}{4}\right). \tag{2.2.151}$$

Note that on $V(x,d,\eta,T)$, $|xv-d^2u|\le 2xT^{\frac{1}{2}+\eta}$, by (2.2.145), and so

$$\left|\frac{\partial H_d}{\partial v}\right| = \left|\frac{1}{u^{3/2}}\cos\left(\frac{\pi(xv-d^2u)^2}{2d^3u}+\frac{3\pi}{4}\right)\cdot\frac{\pi x}{d^3u}(xv-d^2u)\right| = O_x\left(\frac{1}{T^{2-\eta}}\right) \tag{2.2.152}$$

and

$$\left|\frac{\partial H_d}{\partial u}\right| \le \left|\frac{3}{2u^{5/2}}\sin\left(\frac{\pi(xv-d^2u)^2}{2d^3u}+\frac{3\pi}{4}\right)\right|$$
$$+\left|\frac{1}{u^{3/2}}\cos\left(\frac{\pi(xv-d^2u)^2}{2d^3u}+\frac{3\pi}{4}\right)\right|\frac{\pi}{2d^3}\frac{|2(d^2u-xv)d^2u-(d^2u-xv)^2|}{u^2}$$
$$= O_x\left(\frac{1}{T^{2-\eta}}\right). \tag{2.2.153}$$

Using (2.2.152) and (2.2.153), we may replace each sum on the right side of (2.2.150) by a double integral. More precisely, for each $(m,n)\in V(x,d,\eta,T)$,

$$\left|\frac{\sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m}+\frac{3\pi}{4}\right)}{m^{3/2}} - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}H_d(u,v)\,dv\,du\right|$$
$$= \left|\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\{H_d(u,v)-H_d(m,n)\}\,dv\,du\right|$$
$$= O\left(\sup_{\substack{u\in[m-\frac{1}{2},m+\frac{1}{2}]\\ v\in[n-\frac{1}{2},n+\frac{1}{2}]}}\left\{\left|\frac{\partial H_d}{\partial u}(u,v)\right|+\left|\frac{\partial H_d}{\partial v}(u,v)\right|\right\}\right)$$
$$= O_x\left(\frac{1}{T^{2-\eta}}\right). \tag{2.2.154}$$

Adding relations (2.2.154) for all $(m,n) \in V(x,d,\eta,T)$, we see that

$$\sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{3/2}} \sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m}+\frac{3\pi}{4}\right)$$
$$= \sum_{(m,n)\in V(x,d,\eta,T)} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H_d(u,v)dv\,du + O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right). \quad (2.2.155)$$

Let us observe that if we define

$$V^*(x,d,\eta,T) = \cup_{(m,n)\in V(x,d,\eta,T)}[m-\tfrac{1}{2}, m+\tfrac{1}{2}] \times [n-\tfrac{1}{2}, n+\tfrac{1}{2}], \quad (2.2.156)$$

then

$$\text{Area}\,(V(x,d,\eta,T)\backslash V^*(x,d,\eta,T)) \cup (V^*(x,d,\eta,T)\backslash V(x,d,\eta,T)) = O_x(T), \quad (2.2.157)$$

because the perimeter of the trapezoid defining $V(x,d,\eta,T)$ is $O(T)$. Since

$$|H_d(u,v)| = O\left(\frac{1}{T^{3/2}}\right)$$

on $V(x,d,\eta,T) \cup V^*(x,d,\eta,T)$, by (2.2.157), it follows that

$$\sum_{(m,n)\in V(x,d,\eta,T)} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H_d(u,v)dv\,du$$
$$= \int\int_{V(x,d,\eta,T)} H_d(u,v)dv\,du + O_x\left(\frac{1}{T^{1/2}}\right). \quad (2.2.158)$$

Combining (2.2.150) with (2.2.155) and (2.2.158), we find that

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{(-1)^{x/d+1}}{d^{3/2}} \sin\left(\frac{b\sqrt{x}}{d}\right) \int\int_{V(x,d,\eta,T)} H_d(u,v)dv\,du$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \quad (2.2.159)$$

To evaluate the double integrals on the right side of (2.2.159), we perform the change of variable $v = u(w + d^2/x)$ to deduce that

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dvdu = \int_T^{2T} \int_{-1/T^{\frac{1}{2}-\eta}}^{1/T^{\frac{1}{2}-\eta}} H_d(u, u(w+d^2/x))u\,dw\,du$$
$$= \int_T^{2T} \int_{-1/T^{\frac{1}{2}-\eta}}^{1/T^{\frac{1}{2}-\eta}} \frac{\sin\left(\frac{\pi x^2uw^2}{2d^3}+\frac{3\pi}{4}\right)}{\sqrt{u}} dw\,du. \quad (2.2.160)$$

Next, we make a second change of variable to balance the shape of the region of integration by setting $u = Tt$ and $w = z/\sqrt{T}$. Then (2.2.160) reduces to

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \int_1^2 \frac{1}{\sqrt{t}} \int_{-T^\eta}^{T^\eta} \sin\left(\frac{\pi x^2 t z^2}{2d^3} + \frac{3\pi}{4}\right) dz\,dt. \tag{2.2.161}$$

In the inner integral we make a further change of variable, $z = d^{3/2}y/(xt^{1/2})$, so that (2.2.161) now takes the form

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \frac{d^{3/2}}{x} \int_1^2 \frac{1}{t} \int_{-T^\eta x t^{1/2} d^{-3/2}}^{T^\eta x t^{1/2} d^{-3/2}} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy\,dt. \tag{2.2.162}$$

We approximate the inner integral by

$$c_0 := \int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy. \tag{2.2.163}$$

A change of variables followed by an integration by parts yields

$$\begin{aligned}\int_{T^\eta x t^{1/2} d^{-3/2}}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy &= \frac{1}{2}\int_{T^{2\eta}x^2 t d^{-3}}^{\infty} \frac{\sin\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\rho^{1/2}} d\rho \\ &= -\left.\frac{\cos\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\pi\rho^{1/2}}\right|_{T^{2\eta}x^2 t d^{-3}}^{\infty} - \frac{1}{2\pi}\int_{T^{2\eta}x^2 t d^{-3}}^{\infty} \frac{\cos\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\rho^{3/2}} d\rho.\end{aligned} \tag{2.2.164}$$

By (2.2.164), it follows that, uniformly for $1 \le t \le 2$,

$$\left| \int_{T^\eta x t^{1/2} d^{-3/2}}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy \right| = O_x\left(\frac{1}{T^\eta}\right). \tag{2.2.165}$$

It is clear that the same bound as in (2.2.165) also holds for the integral from $-\infty$ to $-T^\eta x t^{1/2} d^{-3/2}$. Using these relations in combination with (2.2.162), we deduce that

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \frac{d^{3/2} c_0 \log 2}{x} + O_x\left(\frac{1}{T^\eta}\right). \tag{2.2.166}$$

We now insert (2.2.166) into the right-hand side of (2.2.159) to deduce that

$$\begin{aligned}U_2(a,b,T,\eta) = x^{-1/4} c_0 \log 2 \sum_{d|x} (-1)^{x/d+1} \sin\left(\frac{b\sqrt{x}}{d}\right) \\ + O_x\left(\frac{1}{T^\eta}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right).\end{aligned} \tag{2.2.167}$$

Recall that $b = \pi\sqrt{x}(1-2\theta)$. Therefore, the series over d on the right-hand side of (2.2.167) cannot cancel for general θ. Thus, in order for the convergence of our initial series $S_1(a,\theta)$ to hold for general θ, it is necessary that c_0 be equal to 0, and indeed it is. To that end [126, p. 435, formula 3.691, no. 1],

$$\begin{aligned} c_0 &= \int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy \\ &= -\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2\right) dy + \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}y^2\right) dy \\ &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0. \end{aligned}$$

In particular, we note that the term $\frac{3\pi}{4}$ in the argument of the sine on the right side of (2.2.163) *is essential* in order to have $c_0 = 0$. We conclude from (2.2.167) that

$$|U_2(a,b,T,\eta)| = O_x\left(\frac{1}{T^{\eta}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.168}$$

By (2.2.168) and (2.2.139),

$$|U_1(a,b,T,\eta)| + |U_2(a,b,T,\eta)| = O_x\left(\frac{1}{T^{\eta/4}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.169}$$

We now let $\eta = \frac{2}{17}$. Then both O-terms on the right-hand side of (2.2.169) are $O_x(1/T^{1/34})$, and so by (2.2.111),

$$\left|\sum_{a,b,T}\right| = O_x\left(\frac{1}{T^{1/34}}\right), \tag{2.2.170}$$

uniformly for $\theta \in [0,1]$, where on the left side of (2.2.170) we deleted δ, which is fixed (recall that $\delta = \eta/39 = 2/663$). With (2.2.170) in hand, the proof of the uniform convergence of the initial series $S_1(a,\theta)$ can immediately be completed, as in the previous case when x was not an integer.

2.2.10 Completion of the Proof of Entry 2.1.1

We return to the function $G(\theta)$ defined in Sect. 2.2.1, which we now know is well-defined and continuous on $[0,1]$. We want to prove that

$$\begin{aligned} &\sin^2(\pi\theta)\left\{\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\frac{1}{2}-\theta\right) + \frac{1}{4}\cot(\pi\theta)\right\} \\ &= \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}\sin^2(\pi\theta) \\ &= G(\theta). \end{aligned} \tag{2.2.171}$$

The identity $G(\theta) = -G(1-\theta)$ is also satisfied. We find the Fourier sine series of $G(\theta)$ on $(0, \frac{1}{2})$, and so write

$$G(\theta) = \sum_{j=1}^{\infty} b_j \sin(2\pi j\theta). \tag{2.2.172}$$

For $j \geq 1$, interchanging the order of integration and double summation by the uniform convergence and continuity established in the foregoing sections, we find that

$$\begin{aligned} b_j &= 2\sqrt{x}\int_0^{1/2} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\ &\qquad \times \sin^2(\pi\theta)\sin(2\pi j\theta)d\theta \\ &= \sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int_0^{1/2} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\ &\qquad \times\Big(\sin(2\pi j\theta) - \frac{1}{2}\sin(2\pi\theta(j+1)) - \frac{1}{2}\sin(2\pi\theta(j-1))\Big) d\theta. \end{aligned} \tag{2.2.173}$$

In the first set of integrals of the series on the far right-hand side of (2.2.173), set

$$u = 4\pi\sqrt{m(n+\theta)x}, \quad \text{so that} \quad \frac{d\theta}{\sqrt{m(n+\theta)}} = \frac{du}{2\pi m\sqrt{x}},$$

and in the second set of integrals of the series, set

$$u = 4\pi\sqrt{m(n+1-\theta)x}, \qquad \text{so that} \qquad \frac{d\theta}{\sqrt{m(n+1-\theta)}} = -\frac{du}{2\pi m\sqrt{x}}.$$

Thus, we find that for each $j \geq 1$,

$$\begin{aligned} &\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int_0^{1/2} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\ &\qquad\qquad \times \sin(2\pi j\theta)d\theta \\ &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2\pi m}\left\{ \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(2\pi j\left(\frac{u^2}{16\pi^2 mx} - n\right)\right)du \right. \\ &\qquad \left. + \int_{4\pi\sqrt{m(n+1)x}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(2\pi j\left(n+1-\frac{u^2}{16\pi^2 mx}\right)\right)du \right\} \\ &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2\pi m}\left\{ \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(\frac{u^2 j}{8\pi mx}\right)du \right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - \int_{4\pi\sqrt{m(n+1)x}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du \Bigg\} \\
&= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2\pi m} \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1)x}} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du \\
&= \sum_{m=1}^{\infty} \frac{1}{2\pi m} \int_0^{\infty} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du. \qquad (2.2.174)
\end{aligned}
$$

Similar calculations hold for the integrals involving $j+1$ and $j-1$ in (2.2.173). Thus, for each $j \geq 1$,

$$
\begin{aligned}
b_j = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \int_0^{\infty} J_1(u) \Bigg\{ &\sin\left(\frac{u^2 j}{8\pi m x}\right) - \frac{1}{2} \sin\left(\frac{u^2 (j+1)}{8\pi m x}\right) \\
&- \frac{1}{2} \sin\left(\frac{u^2 (j-1)}{8\pi m x}\right) \Bigg\} du.
\end{aligned}
$$

For $a, b > 0$, recall the formula [126, p. 759, formula 6.686, no. 5]

$$
\int_0^{\infty} \sin(au^2) J_1(bu) du = \frac{1}{b} \sin\left(\frac{b^2}{4a}\right).
$$

Thus,

$$
b_j = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left\{ \sin\left(\frac{2\pi m x}{j}\right) - \frac{1}{2} \sin\left(\frac{2\pi m x}{j+1}\right) - \frac{1}{2} \sin\left(\frac{2\pi m x}{j-1}\right) \right\}, \tag{2.2.175}
$$

where the last term is not present if $j = 1$. From the fact that for any real number y,

$$
- \sum_{m=1}^{\infty} \frac{\sin(2\pi m y)}{\pi m} = \begin{cases} 0, & \text{if } y \text{ is an integer}, \\ y - [y] - \frac{1}{2}, & \text{if } y \text{ is not an integer}, \end{cases} \tag{2.2.176}
$$

we deduce that

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{\sin(2\pi m x/j)}{\pi m} &= \begin{cases} 0, & \text{if } x/j \text{ is an integer}, \\ -\dfrac{x}{j} + \left[\dfrac{x}{j}\right] + \dfrac{1}{2}, & \text{if } x/j \text{ is not an integer}, \end{cases} \\
&= F\left(\frac{x}{j}\right) - \frac{x}{j} + \frac{1}{2}. \qquad (2.2.177)
\end{aligned}
$$

Hence, from (2.2.175) and (2.2.177), we find that

$$
\begin{aligned}
b_j = \frac{1}{2} \Bigg\{ &F\left(\frac{x}{j}\right) - \frac{x}{j} + \frac{1}{2} - \frac{1}{2} \left(F\left(\frac{x}{j+1}\right) - \frac{x}{j+1} + \frac{1}{2} \right) \\
&- \frac{1}{2} \left(F\left(\frac{x}{j-1}\right) - \frac{x}{j-1} + \frac{1}{2} \right) \Bigg\},
\end{aligned}
$$

where the last term is not present if $j = 1$. Thus,

$$b_1 = \frac{1}{8} - \frac{3x}{8} + \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right), \tag{2.2.178}$$

and for $j \geq 2$,

$$b_j = \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) + \frac{x}{2j(j^2-1)}. \tag{2.2.179}$$

Next, we find the Fourier sine series on $(0, \frac{1}{2})$ of the left-hand side of (2.2.171). We have

$$\begin{aligned} &F\left(\frac{x}{n}\right)\sin(2\pi n\theta)\sin^2(\pi\theta) \\ &\quad = \frac{1}{2}F\left(\frac{x}{n}\right)\left\{\sin(2\pi n\theta) - \frac{1}{2}\sin(2\pi\theta(n+1)) - \frac{1}{2}\sin(2\pi\theta(n-1))\right\} \end{aligned}$$

and

$$\cot(\pi\theta)\sin^2(\pi\theta) = \cos(\pi\theta)\sin(\pi\theta) = \frac{1}{2}\sin(2\pi\theta).$$

Also, since $0 < \theta < 1$, by (2.2.176),

$$\begin{aligned} \sin^2(\pi\theta)\Big(\frac{1}{2} - \theta\Big) &= \frac{1}{2}\Big(1 - \cos(2\pi\theta)\Big)\sum_{m=1}^{\infty}\frac{\sin(2\pi m\theta)}{\pi m} \\ = \frac{1}{2}\sum_{m=1}^{\infty}\frac{\sin(2\pi m\theta)}{\pi m} &- \frac{1}{4}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta(m+1))}{\pi m} - \frac{1}{4}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta(m-1))}{\pi m}. \end{aligned}$$

Thus, if the Fourier sine series of the left-hand side of (2.2.171) is

$$\sum_{j=1}^{\infty} c_j \sin(2\pi j\theta),$$

then

$$c_1 = \frac{1}{8} - \frac{3x}{8} + \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right) = b_1,$$

by (2.2.178), and for $j \geq 2$,

$$c_j = \frac{x}{2j(j^2-1)} + \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) = b_j,$$

by (2.2.179), which completes the proof of (2.1.5).

2.3 Proof of Ramanujan's First Bessel Function Identity (Symmetric Form)

We prove Ramanujan's first Bessel function identity (2.1.5), emphasizing that the double sum on the right-hand side of (2.1.5) is being interpreted symmetrically, i.e., the product mn of the summation indices m and n tends to infinity. A slight modification of the analysis from [26, pp. 354–356], in particular, Lemma 14 of [26], shows that the series on the right-hand side of (2.1.5) converges uniformly with respect to θ on any interval $0 < \theta_1 \leq \theta \leq \theta_2 < 1$. (There is a misprint in (3.5) of Theorem 4 in [26]; read $b(n)/\mu_n^{\sigma-1/2m}$ for $b(n)\mu_n^{\sigma-1/2m}$.) By continuity, it therefore suffices to prove Entry 2.1.1 for rational $\theta = a/q$, where q is prime and $0 < a < q$.

First define

$$\begin{aligned} &H(a,q,x)\\ &:= \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+a/q)x}\right)}{\sqrt{m(n+a/q)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-a/q)x}\right)}{\sqrt{m(n+1-a/q)}}\right\}\\ &= \frac{\sqrt{qx}}{2}\left\{\sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}} - \sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv -a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}}\right\}. \end{aligned}$$

With the restriction $\theta = a/q$ and with the notation above, we now reformulate Entry 2.1.1.

Theorem 2.3.1. *If q is prime and $0 < a < q$, then*

$$H(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right) - \pi x\left(\frac{1}{2}-\frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right) =: P(a,q,x).$$

In the analysis that follows, we demonstrate that in order to prove Theorem 2.3.1, it suffices to prove the next theorem.

Theorem 2.3.2. *Let q be a positive integer, and let χ be an odd primitive character modulo q. Then, for any $x > 0$,*

$$\sum_{n\leq x}{}' d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}) + \frac{i\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}J_1\left(4\pi\sqrt{nx/q}\right). \tag{2.3.1}$$

Proof. Suppose that χ is a primitive nonprincipal odd character modulo q. Then [101, p. 71]

$$\left(\frac{\pi}{q}\right)^{-(2s+1)/2}\Gamma\left(s+\frac{1}{2}\right)L(2s,\chi) = -\frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(1-s)}\Gamma(1-s)L(1-2s,\overline{\chi}). \tag{2.3.2}$$

Recall again the functional equation of $\zeta(s)$, namely,

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-(1/2-s)}\Gamma\Big(\frac{1}{2}-s\Big)\zeta(1-2s). \tag{2.3.3}$$

Multiply (2.3.2) and (2.3.3) to deduce that

$$\begin{aligned}\frac{\pi^{-2s-1/2}}{q^{-s-1/2}}&\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big)\zeta(2s)L(2s,\chi)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-3/2+2s}}{q^{-1+s}}\Gamma(1-s)\Gamma\Big(\frac{1}{2}-s\Big)L(1-2s,\overline{\chi})\zeta(1-2s).\end{aligned} \tag{2.3.4}$$

If we invoke the duplication formula for the gamma function,

$$\Gamma(2s)\sqrt{\pi} = 2^{2s-1}\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big),$$

then (2.3.4) can be written as

$$\begin{aligned}\frac{\pi^{-2s-1/2}}{q^{-s-1/2}}&\frac{\sqrt{\pi}\Gamma(2s)}{2^{2s-1}}\zeta(2s)L(2s,\chi)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-3/2+2s}}{q^{-1+s}}\frac{\Gamma\Big(2(1/2-s)\Big)\sqrt{\pi}}{2^{2(1/2-s)-1}}L(1-2s,\overline{\chi})\zeta(1-2s)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-1+2s}}{q^{-1+s}}\frac{\Gamma(1-2s)}{2^{-2s}}L(1-2s,\overline{\chi})\zeta(1-2s).\end{aligned}$$

Thus,

$$\begin{aligned}\Big(\frac{2\pi}{\sqrt{q}}\Big)^{-2s}&\Gamma(2s)L(2s,\chi)\zeta(2s)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\Big(\frac{2\pi}{\sqrt{q}}\Big)^{2s-1}\Gamma(1-2s)L(1-2s,\overline{\chi})\zeta(1-2s).\end{aligned}$$

Replacing s by $s/2$, we have

$$\Big(\frac{2\pi}{\sqrt{q}}\Big)^{-s}\Gamma(s)L(s,\chi)\zeta(s) = -\frac{i\tau(\chi)}{\sqrt{q}}\Big(\frac{2\pi}{\sqrt{q}}\Big)^{s-1}\Gamma(1-s)L(1-s,\overline{\chi})\zeta(1-s).$$

In the notation of Theorem 2 of [26], $q=0$, $r=m=1$, $\lambda_n=\mu_n=2\pi n/\sqrt{q}$, $a(n)=d_\chi(n)$, $b(n)=-i\tau(\chi)d_{\overline{\chi}}(n)/\sqrt{q}$, and $K_1(2\sqrt{\mu_n x};0;1)=J_1(2\sqrt{\mu_n x})$. We therefore record the following special case of [26, Theorem 2]. Let $x>0$. Then

$$\sum_{\lambda_n\le x}{}' d_\chi(n) = \frac{-i\tau(\chi)}{\sqrt{q}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\left(\frac{x}{\mu_n}\right)^{1/2} J_1(2\sqrt{\mu_n x}) + Q_0(x), \tag{2.3.5}$$

where

$$Q_0(x) = \frac{1}{2\pi i} \int_C \frac{(2\pi/\sqrt{q})^{-s} L(s,\chi)\zeta(s) x^s}{s} ds,$$

where C is a positively oriented closed contour with the singularities of the integrand in the interior.

We now replace x by $2\pi x/\sqrt{q}$ in (2.3.5) to obtain

$$\sum_{n \le x}{}' d_\chi(n) = \frac{-i\tau(\chi)}{\sqrt{q}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \left(\frac{x}{n}\right)^{1/2} J_1(4\pi\sqrt{nx/q}) + Q_0(2\pi x/\sqrt{q}). \quad (2.3.6)$$

Now, since $\zeta(0) = -\frac{1}{2}$,

$$Q_0(2\pi x/\sqrt{q}) = \frac{1}{2\pi i} \int_C \frac{L(s,\chi)\zeta(s)x^s}{s} ds = -\frac{1}{2} L(0,\chi) + L(1,\chi)x. \quad (2.3.7)$$

From the functional equation (2.3.2),

$$\left(\frac{\pi}{q}\right)^{-1/2} \Gamma(1/2) L(0,\chi) = -i \frac{\tau(\chi)}{\sqrt{q}} \frac{q}{\pi} L(1,\overline{\chi}).$$

So,

$$L(0,\chi) = -\frac{i\tau(\chi)}{\pi} L(1,\overline{\chi}).$$

Thus, from (2.3.7),

$$Q_0(2\pi x/\sqrt{q}) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi} L(1,\overline{\chi}). \quad (2.3.8)$$

Lastly, putting (2.3.8) in (2.3.6) and using the identity $\tau(\chi)\tau(\overline{\chi}) = -q$, since χ is odd, we complete the proof of Theorem 2.3.2. □

After proving the following lemma, we show that Theorem 2.3.2 implies Theorem 2.3.1.

Lemma 2.3.1. *If* $0 < a < q$ *and* $(a,q) = 1$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi na}{q}\right) = -i \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \sum_{1 \le n \le dx/q}{}' d_\chi(n).$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi na}{q}\right) &= \sum_{d|q} \sum_{(n,q)=q/d} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi na}{q}\right) \\ &= \sum_{d|q} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sin\left(\frac{2\pi ma}{d}\right)\end{aligned}$$

$$= \sum_{m=1}^{\infty} F\Big(\frac{x}{qm}\Big) + \sum_{\substack{d|q\\d>1}} \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big) \sin\Big(\frac{2\pi ma}{d}\Big)$$

$$= \sum_{m=1}^{\infty} F\Big(\frac{x}{qm}\Big) + \frac{1}{2}\sum_{\substack{d|q\\d>1}} \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big)\big(e^{2\pi ima/d} - e^{-2\pi ima/d}\big). \tag{2.3.9}$$

We know that for any positive integers a_1, a_2, and q,

$$\sum_{\chi \bmod q} \chi(a_1)\overline{\chi}(a_2) = \begin{cases} \phi(q), & \text{if } a_1 \equiv a_2 \pmod q \text{ and } (a_1, q) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.10}$$

Using (2.3.10) and the formula [101, p. 65]

$$\chi(n)\tau(\overline{\chi}) = \sum_{h=1}^{q} \overline{\chi}(h) e^{2\pi inh/q}, \tag{2.3.11}$$

for any character χ modulo q, we find that for m, d such that $(m, d) = 1$ and $d > 1$,

$$\begin{aligned} e^{2\pi ima/d} &= \frac{1}{\phi(d)} \sum_{h=1}^{d} e^{2\pi imh/d} \sum_{\chi \bmod d} \chi(a)\overline{\chi}(h) \\ &= \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a) \sum_{h=1}^{d} \overline{\chi}(h) e^{2\pi imh/d} \\ &= \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a)\tau(\overline{\chi})\chi(m). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\sum_{\substack{d|q\\d>1}} \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big)\big(e^{2\pi ima/d} - e^{-2\pi ima/d}\big) \\ &\qquad = \sum_{\substack{d|q\\d>1}} \frac{1}{2\phi(d)} \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big) \sum_{\chi \bmod d} \chi(a)\tau(\overline{\chi})(\chi(m) - \chi(-m)) \\ &\qquad = \sum_{\substack{d|q\\d>1}} \frac{1}{\phi(d)} \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big) \sum_{\substack{\chi \bmod d\\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi})\chi(m) \\ &\qquad = \sum_{\substack{d|q\\d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) \sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\Big(\frac{dx}{qm}\Big)\chi(m) \end{aligned}$$

$$= \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) \sum_{m=1}^{\infty} F\Big(\frac{dx}{qm}\Big)\chi(m),$$

since $\chi(m) = 0$ if $(m,d) > 1$. Hence, using the calculation above in (2.3.9), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} F\Big(\frac{x}{n}\Big) \sin\Big(\frac{2\pi n a}{q}\Big) \\
&\quad = \sum_{m=1}^{\infty} F\Big(\frac{x}{qm}\Big) + \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \sum_{m=1}^{\infty} F\Big(\frac{dx}{qm}\Big)\chi(m) \\
&\quad = {\sum_{1 \le n \le x/q}}' d(n) + \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) {\sum_{1 \le n \le dx/q}}' d_{\chi}(n),
\end{aligned}$$

where we used (2.1.12). Thus, our proof of Lemma 2.3.1 is complete. □

As promised, we now show that Theorem 2.3.2 implies Theorem 2.3.1.

Proof of Theorem 2.3.1. We easily see that $H(a,q,x) = -H(q-a,q,x)$ and $P(a,q,x) = -P(q-a,q,x)$, and so we can assume that $0 < a < q/2$. Consider

$$\begin{aligned}
&H(a,q,x) \\
&\quad = \frac{\sqrt{x}}{2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\big(4\pi\sqrt{m(n+a/q)x}\big)}{\sqrt{m(n+a/q)}} - \frac{J_1\big(4\pi\sqrt{m(n+1-a/q)x}\big)}{\sqrt{m(n+1-a/q)}} \right\} \\
&\quad = \frac{\sqrt{qx}}{2} \sum_{m=1}^{\infty} \left\{ \sum_{\substack{r=1 \\ r \equiv a \bmod q}}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} - \sum_{\substack{r=1 \\ r \equiv -a \bmod q}}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \right\} \\
&\quad = \frac{\sqrt{qx}}{2\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \sum_{\chi \bmod q} \overline{\chi}(r)\big(\chi(a) - \chi(-a)\big) \\
&\quad = \frac{\sqrt{qx}}{\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \overline{\chi}(r)\chi(a) \\
&\quad = \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \overline{\chi}(r) \sqrt{\frac{x}{qmr}} J_1\big(4\pi\sqrt{mrx/q}\big) \\
&\quad = \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} J_1\big(4\pi\sqrt{nx/q}\big). \qquad (2.3.12)
\end{aligned}$$

On the other hand, by Lemma 2.3.1,

$$P(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi na}{q}\right) - \pi x\left(\frac{1}{2} - \frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right)$$
$$= \frac{-i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) {\sum_{1\le n\le x}}' d_{\chi}(n) - \pi x\left(\frac{1}{2} - \frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right).$$

Applying Theorem 2.3.2 and using (2.3.12), we only need to show that

$$\frac{i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi})\left(L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi})\right) = -\pi x\left(\frac{1}{2} - \frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right). \tag{2.3.13}$$

We use the following formulas, which are (2.5) and (2.8) in [71]:

$$\tau(\overline{\chi})L(1,\chi) = 2\pi i \sum_{1\le h<q/2} \overline{\chi}(h)\left(\frac{1}{2} - \frac{h}{q}\right), \tag{2.3.14}$$

$$\tau(\chi)L(1,\overline{\chi}) = -\frac{\pi}{\tau(\overline{\chi})} \sum_{1\le h<q/2} \overline{\chi}(h)\cot\left(\frac{\pi h}{q}\right). \tag{2.3.15}$$

We also can easily deduce from (2.3.10) that

$$\sum_{\chi \text{ even}} \chi(a)\overline{\chi}(h) = \sum_{\chi \text{ odd}} \chi(a)\overline{\chi}(h) = \begin{cases} \phi(q)/2, & \text{if } h \equiv a \pmod{q}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3.16}$$

since $(a,q) = 1$.

Then, using (2.3.14)–(2.3.16), we deduce that

$$\frac{i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi})\left(L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi})\right)$$
$$= \left\{-\frac{2\pi x}{\phi(q)} \sum_{1\le h<q/2} \left(\frac{1}{2} - \frac{h}{q}\right) + \frac{1}{2\phi(q)} \sum_{1\le h<q/2} \cot\left(\frac{\pi h}{q}\right)\right\} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\overline{\chi}(h)$$
$$= -\pi x\left(\frac{1}{2} - \frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right),$$

which completes the proof of (2.3.13) and therefore also of Theorem 2.3.1. □

In fact, Theorem 2.3.1 is equivalent to the following theorem [57].

Theorem 2.3.3. *Let q be a positive integer, and let χ be an odd primitive character modulo q. Then, for any $x>0$,*

$$\sum_{n\le x}{}' d_{\chi}(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}) + \frac{i\sqrt{x}}{\tau(\overline{\chi})}\sum_{1\le h<q/2}\overline{\chi}(h)$$
$$\times \lim_{N\to\infty}\sum_{mn\le N}\left\{\frac{J_1\left(4\pi\sqrt{m\,(n+h/q)\,x}\right)}{\sqrt{m\,(n+h/q)}} - \frac{J_1\left(4\pi\sqrt{m\,(n+1-h/q)\,x}\right)}{\sqrt{m\,(n+1-h/q)}}\right\}. \tag{2.3.17}$$

2.4 Proof of Ramanujan's Second Bessel Function Identity (with the Order of Summation Reversed)

2.4.1 Preliminary Results

We now embark on a proof of Entry 2.1.2, where now we consider the double series on the right side of (2.1.6) to be an iterated double sum. As emphasized in the introduction, we will approach Entry 2.1.2 with the order of summation on the double series reversed. Our proof depends upon the following formulation of the Poisson summation formula due to A.P. Guinand [132, p. 595].

Theorem 2.4.1. *If $f(x)$ can be represented as a Fourier integral, $f(x)$ tends to 0 as $x\to\infty$, and $xf'(x)\in L^p(0,\infty)$ for some p, $1<p\le 2$, then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N} f(n) - \int_0^N f(t)\,dt\right\} = \lim_{N\to\infty}\left\{\sum_{n=1}^{N} g(n) - \int_0^N g(t)\,dt\right\}, \tag{2.4.1}$$

where

$$g(x) := 2\int_0^\infty f(t)\cos(2\pi xt)\,dt.$$

We need the following two lemmas from [48, Lemmas 3.5, 3.4].

Lemma 2.4.1. *We have*

$$\int_0^\infty I_1(x)dx = 0.$$

Lemma 2.4.2. *With I_ν defined by (2.1.7) and $b,c>0$,*

$$\int_0^\infty \cos(bx^2)I_1(cx)dx = \frac{1}{c}\sin\left(\frac{c^2}{4b}\right). \tag{2.4.2}$$

2.4.2 Reformulation of Entry 2.1.2

Theorem 2.4.2. *Let $F(x)$ be defined by* (2.1.4) *and let $I_1(x)$ be defined by* (2.1.7). *Then, for $x > 0$ and $0 < \theta < 1$,*

$$\frac{1}{2}\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}$$
$$=\frac{1}{2\pi}\left(\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right)-\int_0^M\sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}\right.$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+1-\theta)x}{m}\right)\right.$$
$$\left.\left.-\int_0^M\sin\left(\frac{2\pi(n+1-\theta)x}{t}\right)dt\right\}\right). \qquad (2.4.3)$$

Proof. Let

$$f(t)=\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}$$

in Theorem 2.4.1. First, setting $u = 4\pi\sqrt{t(n+\theta)x}$ and using Lemma 2.4.1, we find that

$$\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}-\int_0^M\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}dt\right\}$$
$$=\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}\right\}-\frac{1}{2\pi(n+\theta)\sqrt{x}}\int_0^\infty I_1(u)du$$
$$=\sum_{m=1}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}. \qquad (2.4.4)$$

Second, putting $u = 4\pi\sqrt{t(n+\theta)x}$ and using Lemma 2.4.2, we find that

$$g(m)=2\int_0^\infty\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}\cos(2\pi mt)dt$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\int_0^\infty I_1(u)\cos\left(\frac{mu^2}{8\pi(n+\theta)x}\right)du$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\sin\left(\frac{2\pi(n+\theta)x}{m}\right). \qquad (2.4.5)$$

Hence,

$$\lim_{M\to\infty}\left\{\sum_{m=1}^{M} g(m) - \int_0^M g(t)\,dt\right\}$$
$$= \frac{1}{\pi(n+\theta)\sqrt{x}} \lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}. \tag{2.4.6}$$

We make a digression here to demonstrate conclusively that the limit in (2.4.6) actually does exist. Write, for $a>0$,

$$\begin{aligned}
&\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a}{m}\right) - \int_0^M \sin\left(\frac{a}{t}\right)dt\right\}\\
&\quad= \lim_{M\to\infty}\left\{\sum_{m=1}^{M}\left(\sin\left(\frac{a}{m}\right)-\frac{a}{m}+\frac{a}{m}\right) - \int_1^M\left(\sin\left(\frac{a}{t}\right)-\frac{a}{t}+\frac{a}{t}\right)dt\right\}\\
&\qquad - \int_0^1 \sin\left(\frac{a}{t}\right)dt\\
&\quad= L_1 - L_2 + \lim_{M\to\infty}\left\{a\sum_{m=1}^{M}\frac{1}{m} - a\int_1^M\frac{dt}{t}\right\} - \int_0^1\sin\left(\frac{a}{t}\right)dt\\
&\quad= L_1 - L_2 - \int_0^1\sin\left(\frac{a}{t}\right)dt + a\left\{\log M + \gamma + o(1) - \log M\right\}\\
&\quad= L_1 - L_2 - \int_0^1\sin\left(\frac{a}{t}\right)dt + a\gamma,
\end{aligned}$$

where γ denotes Euler's constant and where

$$\begin{aligned}
L_1 &= \lim_{M\to\infty}\sum_{m=1}^{M}\left(\sin\left(\frac{a}{m}\right)-\frac{a}{m}\right),\\
L_2 &= \lim_{M\to\infty}\int_1^M\left(\sin\left(\frac{a}{t}\right)-\frac{a}{t}\right)dt.
\end{aligned}$$

Returning to our proof and putting together (2.4.4) and (2.4.6) in (2.4.1), we find that

$$\sum_{m=1}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}$$
$$= \frac{1}{\pi(n+\theta)\sqrt{x}} \lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}. \tag{2.4.7}$$

Now in (2.4.7) replace θ by $1-\theta$ and add the result to (2.4.7). Sum both sides on n, $0 \le n < \infty$. Then multiply the resulting equality by $\frac{1}{2}\sqrt{x}$ to deduce (2.4.3) and thus complete the proof of Theorem 2.4.2. □

If we compare (2.1.6) with (2.4.3), we see that in order to prove Entry 2.1.2, *but with the order of summation reversed in the double series*, we need to prove that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) - \frac{1}{4} + x\log(2\sin(\pi\theta))$$
$$= \frac{1}{2\pi}\left(\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}\right.$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+1-\theta)x}{m}\right)\right.$$
$$\left.\left.-\int_0^M \sin\left(\frac{2\pi(n+1-\theta)x}{t}\right)dt\right\}\right).$$

2.4.3 The Convergence of (2.4.3)

Fix $x > 0$, and set $a = 2\pi x$. We are interested in the question of convergence (pointwise, or uniformly with respect to θ on compact subintervals of the interval $(0,1)$) of the series

$$S(a,\theta) := \sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^M \sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a(n+1-\theta)}{m}\right)\right.$$
$$\left.-\int_0^M \sin\left(\frac{a(n+1-\theta)}{t}\right)dt\right\}.$$

For $m > 2$,

$$\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m}\sin\left(\frac{a(n+\theta)}{t}\right)dt$$
$$= \int_{m-1}^{m}\left(\sin\left(\frac{a(n+\theta)}{m}\right) - \sin\left(\frac{a(n+\theta)}{t}\right)\right)dt$$
$$= \int_{m-1}^{m} 2\sin\frac{1}{2}\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)}{t}\right)\cos\frac{1}{2}\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)}{t}\right)dt.$$

Thus,

$$\begin{aligned}
&\left|\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right| \\
&\quad \leq 2\int_{m-1}^{m} \left|\sin\left(\frac{a(n+\theta)(t-m)}{2mt}\right)\right| dt \\
&\quad \leq \int_{m-1}^{m} \frac{a(n+\theta)(m-t)}{mt} dt < \frac{a(n+\theta)}{m(m-1)}.
\end{aligned} \tag{2.4.8}$$

Fix $\delta_1 > 0$ and set $M_1 = [n^{1+\delta_1}]$, where $[x]$ denotes the greatest integer $\leq x$. We write

$$\begin{aligned}
&\lim_{M\to\infty} \left\{ \sum_{m=1}^{M} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^M \sin\left(\frac{a(n+\theta)}{t}\right) dt \right\} \\
&= \sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt \\
&\quad + \lim_{M\to\infty} \left\{ \sum_{m=M_1+1}^{M} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_{M_1}^{M} \sin\left(\frac{a(n+\theta)}{t}\right) dt \right\}.
\end{aligned}$$

Here the last limit exists, and, by (2.4.8), is a real number bounded by

$$\left|\sum_{m=M_1+1}^{\infty} \frac{a(n+\theta)}{m(m-1)}\right| = \frac{a(n+\theta)}{M_1} \ll_a \frac{1}{n^{\delta_1}},$$

uniformly with respect to θ in $[0,1]$. Therefore the series

$$\sum_{n=0}^{\infty} \frac{1}{n+\theta} \lim_{M\to\infty} \left\{ \sum_{m=M_1+1}^{M} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_{M_1}^{M} \sin\left(\frac{a(n+\theta)}{t}\right) dt \right\}$$

converges uniformly with respect to θ, and the same holds for the other, similar series involving $n+1-\theta$. We deduce that the series

$$\begin{aligned}
S_1(a,\theta,\delta_1) &:= \sum_{n=0}^{\infty} \frac{1}{n+\theta} \left\{ \sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt \right\} \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{n+1-\theta} \left\{ \sum_{m=1}^{M_1} \sin\left(\frac{a(n+1-\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+1-\theta)}{t}\right) dt \right\}
\end{aligned}$$

converges pointwise if and only if the initial sum $S(a,\theta)$ converges pointwise, and $S_1(a,\theta,\delta_1)$ converges uniformly with respect to θ on compact subintervals of $(0,1)$ if and only if this holds for $S(a,\theta)$.

Next, we need a bound for

$$\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt.$$

We write this expression in the form

$$\sum_{m=1}^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{t}\right) dt$$
$$+ \sum_{m=[\sqrt{n}]+1}^{M_1} \left(\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right).$$

Here the first sum is bounded in absolute value by $\sqrt{n}$. The same bound holds for the integral, i.e.,

$$\left|\int_0^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right| < \sqrt{n}.$$

As for the last sum above, we use (2.4.8) to bound each term in order to conclude that

$$\sum_{m=[\sqrt{n}]+1}^{M_1} \left(\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right)$$
$$\ll \sum_{m=[\sqrt{n}]+1}^{M_1} \frac{a(n+\theta)}{m(m-1)} < \frac{a(n+\theta)}{[\sqrt{n}]}.$$

We thus have shown that

$$\left|\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right| \ll_a \sqrt{n},$$

uniformly with respect to θ on compact subsets of $(0,1)$.

With this bound in hand, we now proceed to remove the dependence on θ from the coefficients $1/(n+\theta)$ and $1/(n+1-\theta)$ in $S_1(a,\theta,\delta_1)$. More specifically, we consider the sum

$$S_2(a,\theta,\delta_1) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left\{\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right.$$
$$\left. + \sum_{m=1}^{M_1} \sin\left(\frac{a(n+1-\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+1-\theta)}{t}\right) dt\right\}.$$

Note that the sum

$$\sum_{n=0}^{\infty}\left\{\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+\theta}\right)\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_0^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}\right.$$
$$+\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+1-\theta}\right)\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+1-\theta)}{m}\right)\right.$$
$$\left.\left.-\int_0^{M_1}\sin\left(\frac{a(n+1-\theta)}{t}\right)dt\right\}\right\} \qquad (2.4.9)$$

is uniformly and absolutely convergent, since for each n,

$$\left|\frac{1}{n+\frac{1}{2}}-\frac{1}{n+\theta}\right|\left|\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_0^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right|$$
$$\ll_a \frac{|\theta-\frac{1}{2}|}{(n+\frac{1}{2})(n+\theta)}\sqrt{n}\ll_a \frac{1}{n^{3/2}},$$

uniformly in θ. We obtain the same bound for the other sum in (2.4.9) by the same argument. It follows that the sum $S_2(a,\theta,\delta_1)$ is convergent for a given value of θ if and only if $S_1(a,\theta,\delta_1)$ is convergent for that value of θ. Also, $S_2(a,\theta,\delta_1)$ is uniformly convergent with respect to θ on closed subintervals of $(0,1)$ if and only if $S_1(a,\theta,\delta_1)$ has this property. Next, using the oscillatory behavior of the function $y\mapsto\sin y$, we perform another truncation of the inner sum in $S_2(a,\theta,\delta_1)$, by replacing M_1 by a smaller value M_2, to be determined later. Consider the sum

$$S_3(a,\theta):=\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left\{\sum_{m=1}^{M_2}\left(\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\right.$$
$$\left.-\int_0^{M_2+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt\right\}.$$

In order to relate the convergence of $S_3(a,\theta)$ to that of $S_2(a,\theta,\delta_1)$, we estimate, for each $m\in\{M_2+1,M_2+2,\dots,M_1\}$, the quantity

$$\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)$$
$$-\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{m}\right)-\sin\left(\frac{a(n+\theta)}{m+u}\right)\right.$$
$$\left.+\sin\left(\frac{a(n+1-\theta)}{m}\right)-\sin\left(\frac{a(n+1-\theta)}{m+u}\right)\right)du.$$

Here,

$$\frac{a(n+\theta)}{m+u} = \frac{a(n+\theta)}{m(1+u/m)} = \frac{a(n+\theta)}{m}\left(1-\frac{u}{m}+O\left(\frac{1}{m^2}\right)\right)$$
$$= \frac{a(n+\theta)}{m} - \frac{a(n+\theta)u}{m^2} + O_a\left(\frac{n}{m^3}\right),$$

uniformly in θ. So,

$$\sin\left(\frac{a(n+\theta)}{m+u}\right) = \sin\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)u}{m^2}\right) + O_a\left(\frac{n}{m^3}\right).$$

We will choose M_2 much larger than $\sqrt{n}$. Then the ratio $a(n+\theta)u/m^2$ will be small, a is fixed, $\theta \in [0,1]$, and $u \in [-\frac{1}{2}, \frac{1}{2}]$. Then, using the estimate

$$\sin(\alpha - \epsilon) = \sin\alpha - \epsilon\cos\alpha + O(\epsilon^2)$$

with $\alpha = a(n+\theta)/m$ and $\epsilon = a(n+\theta)u/m^2$, we see that

$$\sin\left(\frac{a(n+\theta)}{m+u}\right) = \sin\left(\frac{a(n+\theta)}{m}\right) - \frac{a(n+\theta)u}{m^2}\cos\left(\frac{a(n+\theta)}{m}\right)$$
$$+ O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).$$

Since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{a(n+\theta)u}{m^2}\cos\left(\frac{a(n+\theta)}{m}\right) du = 0,$$

it follows that

$$\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \sin\left(\frac{a(n+\theta)}{t}\right) dt = O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).$$

Similarly,

$$\sin\left(\frac{a(n+1-\theta)}{m}\right) - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \sin\left(\frac{a(n+1-\theta)}{t}\right) dt = O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).$$

We add up these relations for $m = M_2+1, \dots, M_1$ to find that

$$\sum_{m=M_2+1}^{M_1}\left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right)$$
$$- \int_{M_2+\frac{1}{2}}^{M_1+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt$$
$$= O\left(\frac{n^2}{M_2^3}\right) + O\left(\frac{n}{M_2^2}\right),$$

uniformly for θ in compact subsets of $(0,1)$. Therefore, if we choose, for instance, $M_2 = [n^{2/3}\log n]$, then the series

$$\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left(\sum_{m=M_2+1}^{M_1} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right) - \int_{M_2+\frac{1}{2}}^{M_1+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt \right)$$

is uniformly and absolutely convergent.

Let us also remark that for $t \in [M_1, M_1+\frac{1}{2}]$,

$$\frac{a(n+\theta)}{t} = O\left(\frac{1}{n^{\delta_1}}\right), \qquad \text{and so} \qquad \sin\left(\frac{a(n+\theta)}{t}\right) = O\left(\frac{1}{n^{\delta_1}}\right),$$

and also

$$\int_{M_1}^{M_1+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt = O\left(\frac{1}{n^{\delta_1}}\right). \tag{2.4.10}$$

Hence, the series

$$\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \int_{M_1}^{M_1+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt$$

is uniformly and absolutely convergent. Combining all of the above, we deduce that the initial series $S(a,\theta)$ is convergent for a given value of θ if and only if the series $S_3(a,\theta)$ is convergent for that value of θ. Moreover, $S(a,\theta)$ converges uniformly on compact subintervals of $(0,1)$ if and only if the same holds for $S_3(a,\theta)$.

Let us observe that the contribution of the integrals in (2.4.10) is small, while on the other hand, we do not have any cancellation inside the integrals

$$\int_{M_2}^{M_2+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt.$$

Indeed, one can show that the integrand here is almost constant, in fact equal to

$$2\sin\left(\frac{an}{M_2}\right) + O\left(\frac{1}{n^{1/3}\log^2 n}\right) = \sin\left(\frac{an^{1/3}}{\log^2 n}\right) + O\left(\frac{1}{n^{1/3}\log^2 n}\right).$$

Moreover, one can show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{an^{1/3}}{\log^2 n}\right)$$

is *not* absolutely convergent. This forces us to keep at this stage $M_2 + \frac{1}{2}$ instead of M_2 as the upper limit of integration in the definition of $S_3(a,\theta)$. As a side remark, one can show that the series above, although not absolutely convergent, *is* convergent, via proving that the fractional parts

$$\left\{ \frac{an^{1/3}}{\pi \log^2 n} \right\}$$

are "very" uniformly distributed in the interval $[0,1]$, where "very" means that the discrepancy of the first N terms is $\ll N^{-c}$ for some absolute constant $c > 0$.

Next, we choose a new (integral) parameter M_3, whose precise value as a function of n will be given later, and consider the sum

$$\begin{aligned} S_4(a,\theta) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \Bigg\{ & \sum_{m=1}^{M_3} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right) \\ & - \int_0^{M_3+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt \\ & + 2 \sum_{m=M_3+1}^{M_2} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) - 2\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{t}\right) dt \Bigg\} . \end{aligned}$$

Note that the sum $S_4(a,\theta)$ differs from $S_3(a,\theta)$ by having θ replaced by $\frac{1}{2}$ in the range $M_3 + 1 \le m \le M_2$. In order to relate the convergence of these two sums, we write, for $m = M_3 + 1, \ldots, M_2$,

$$\begin{aligned} & \sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ & = 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right)\cos\left(\frac{a(\theta-\frac{1}{2})}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ & = -4\sin\left(\frac{a(n+\frac{1}{2})}{m}\right)\sin^2\left(\frac{a(\theta-\frac{1}{2})}{2m}\right) . \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{m=M_3+1}^{M_2} \left| \sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \right| \\ & \le 4 \sum_{m=M_3+1}^{M_2} \sin^2\left(\frac{a(\theta-\frac{1}{2})}{2m}\right) \ll_a \sum_{m=M_3+1}^{M_2} \frac{1}{m^2} \ll \frac{1}{M_3}, \end{aligned}$$

uniformly with respect to θ. Similarly,

$$\left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\right) dt\right|$$
$$= 4\left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\sin^2\left(\frac{a(n+\frac{1}{2})}{2t}\right) dt\right| \ll_a \int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \frac{dt}{t^2} \ll \frac{1}{M_3}.$$

If we now take $M_3 = [\log^2 n]$, the sum

$$\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left\{ \sum_{m=M_3+1}^{M_2} \left| \sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) \right.\right.$$
$$\left. -2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right)\right| - \left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right)\right.\right.$$
$$\left.\left.\left. + \sin\left(\frac{a(n+1-\theta)}{t}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\right) dt\right|\right\}$$

will be uniformly convergent with respect to θ. Consequently, the sum $S_3(a,\theta)$ will be convergent for a given θ if and only if the sum $S_4(a,\theta)$ converges for the same value of θ, and $S_3(a,\theta)$ converges uniformly on compact subintervals of $(0,1)$ if and only if $S_4(a,\theta)$ does.

In what follows, we define

$$S_5(a,\theta) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left\{ \sum_{m=1}^{M_3} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\right.$$
$$\left. - \int_0^{M_3+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt \right\}$$

and

$$S_6(a) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left\{ \sum_{m=M_3+1}^{M_2} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) - \int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{t}\right) dt \right\},$$

so that

$$S_4(a,\theta) = S_5(a,\theta) + 2S_6(a).$$

Here the inner sum in $S_5(a,\theta)$ has a very short range, of the size of $\log^2 n$, while the inner sum in $S_6(a)$ has a larger range, but is independent of θ. We now turn our attention to $S_5(a,\theta)$ and see whether this sum is pointwise convergent, respectively uniformly convergent on compact subintervals of $(0,1)$. Set

$$A(a,\theta,N) := \sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \sum_{m=1}^{M_3} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right)$$

and

$$B(a,\theta,N) := \sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \int_0^{M_3+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt.$$

Then $S_5(a,\theta)$ converges (respectively converges uniformly on compact subintervals of $(0,1)$), provided that for every $\epsilon > 0$, there exists an $N(\epsilon)$ such that for every $N_1, N_2 > N(\epsilon)$,

$$|A(a,\theta,N_1) + B(a,\theta,N_1) - A(a,\theta,N_2) - B(a,\theta,N_2)| < \epsilon$$

(respectively uniformly for all θ in a given compact subinterval of $(0,1)$).

Fix $\epsilon > 0$. For every positive integer N, we put $A(a,\theta,N)$ in the form

$$A(a,\theta,N) = 2\sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \sum_{1\le m\le \log^2 n} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \cos\left(\frac{a(2\theta-1)}{2m}\right).$$

Here the condition $m \le \log^2 n$ is equivalent to $e^{\sqrt{m}} \le n$. Thus, interchanging the order of summation above, we find that

$$\begin{aligned} A(a,\theta,N) &= 2 \sum_{1\le m\le \log^2 N} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ &= 4 \sum_{1\le m\le \log^2 N} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right). \end{aligned}$$

For two large positive integers $N_1 < N_2$, we put $A(a,\theta,N_2) - A(a,\theta,N_1)$ in the form

$$\begin{aligned} &A(a,\theta,N_2) - A(a,\theta,N_1) \\ &= 4 \sum_{1\le m\le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{N_1+1\le n\le N_2} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right) \\ &\quad + 4 \sum_{\log^2 N_1 < m \le \log^2 N_2} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N_2} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right). \end{aligned}$$

For every positive real numbers $U < V$, consider the function

$$h_{U,V}(y) := \sum_{U\le n\le V} \frac{\sin\{(2n+1)y\}}{2n+1}.$$

With this notation, we may write

$$A(a,\theta,N_2) - A(a,\theta,N_1) = 4 \sum_{1\le m\le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right)$$
$$+ 4 \sum_{\log^2 N_1<m\le \log^2 N_2} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right). \tag{2.4.11}$$

We are interested in the behavior of the function $h_{U,V}(y)$. This function is odd and periodic modulo 2π, and so it is sufficient to study the function on the interval $[0,\pi]$. Also, we note that $h_{U,V}(y) = h_{U,V}(\pi - y)$, and so furthermore, it is sufficient to consider this function on the interval $[0, \frac{1}{2}\pi]$. Observe that $h_{U,V}(0) = 0$. Next, since the series is alternating with decreasing terms,

$$\left|h_{U,V}(\tfrac{1}{2}\pi)\right| = \left|\sum_{U\le n\le V} \frac{(-1)^n}{2n+1}\right| \le \frac{1}{2U+1}.$$

For $0 < y < \frac{1}{2}\pi$, we write $h_{U,V}(y)$ in the form

$$h_{U,V}(y) = h_{U,V}(\tfrac{1}{2}\pi) + h_{U,V}(y) - h_{U,V}(\tfrac{1}{2}\pi) = h_{U,V}(\tfrac{1}{2}\pi) - \int_y^{\frac{1}{2}\pi} h'_{U,V}(t)dt. \tag{2.4.12}$$

Here we write [126, p. 36, formula 1.342, no. 4]

$$h'_{U,V}(t) = \sum_{U\le n\le V} \cos\{(2n+1)t\} = \frac{1}{2\sin t}\left(\sin\{2(\lfloor V\rfloor+1)t\} - \sin(2\lceil U\rceil t)\right), \tag{2.4.13}$$

where $\lfloor V\rfloor$ is the floor of V, that is, the largest integer $\le V$, and $\lceil U\rceil$ is the ceiling of U, that is, the smallest integer $\ge U$. From (2.4.12) and (2.4.13) and an integration by parts,

$$\begin{aligned}
h_{U,V}(y) &= h_{U,V}(\tfrac{1}{2}\pi) - \int_y^{\frac{1}{2}\pi} \frac{1}{2\sin t}\left(\sin\{2(\lfloor V\rfloor+1)t\} - \sin(2\lceil U\rceil t)\right) dt \\
&= h_{U,V}(\tfrac{1}{2}\pi) + \frac{1}{2\sin t}\left(\frac{\cos\{2(\lfloor V\rfloor+1)t\}}{2(\lfloor V\rfloor+1)} - \frac{\cos(2\lceil U\rceil t)}{2\lceil U\rceil}\right)\Bigg|_y^{\frac{1}{2}\pi} \\
&\quad + \int_y^{\frac{1}{2}\pi} \frac{\cos t}{2\sin^2 t}\left(\frac{\cos\{2(\lfloor V\rfloor+1)t\}}{2(\lfloor V\rfloor+1)} - \frac{\cos(2\lceil U\rceil t)}{2\lceil U\rceil}\right) dt \\
&= O\left(\frac{1}{U}\right) + O\left(\frac{1}{Uy}\right) + O\left(\frac{1}{Uy^2}\right) \\
&= O\left(\frac{1}{U}\left(1+\frac{1}{y^2}\right)\right),
\end{aligned}$$

uniformly for $0 < y \le \frac{1}{2}\pi$. If we need a bound that holds for all $y > 0$, we may write

$$|h_{U,V}(y)| = O\left(\frac{1}{U} \cdot \frac{1}{\|y/\pi\|^2}\right),$$

where $\|y/\pi\|$ denotes the distance from y/π to the nearest integer, which is proportional (via a factor of π) to the distance from y to the set $\pi\mathbb{Z} = \{\dots, -\pi, 0, \pi, 2\pi, \dots\}$. Recall that at these points $\pi\mathbb{Z}$, the function $h_{U,V}(y)$ vanishes.

We are now ready to apply these considerations to our expression for $A(a,\theta,N_2) - A(a,\theta,N_1)$ from (2.4.11). For $\log^2 N_1 < m \le \log^2 N_2$ and a fixed, $a/(2m)$ is a small positive number, which belongs to $(0, \frac{1}{2}\pi)$. Hence,

$$\left|h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| = O\left(\frac{1}{e^{\sqrt{m}}}\left(1 + \frac{4m^2}{a^2}\right)\right) = O\left(\frac{m^2}{e^{\sqrt{m}}}\right).$$

It follows that

$$\begin{aligned}
4&\left|\sum_{\log^2 N_1 < m \le \log^2 N_2} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| \\
&\le 4 \sum_{\log^2 N_1 < m \le \log^2 N_2} \left|h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| \\
&= O\left(\sum_{\log^2 N_1 < m \le \log^2 N_2} \frac{m^2}{e^{\sqrt{m}}}\right) \\
&= O\left(\int_{\log^2 N_1}^{\infty} \frac{x^2}{e^{\sqrt{x}}}dx\right) = O\left(\int_{\log N_1}^{\infty} \frac{2t^5}{e^t}dt\right) = O\left(\frac{\log^5 N_1}{N_1}\right).
\end{aligned}$$

Next, we similarly examine the sum

$$4 \sum_{1 \le m \le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right),$$

at least as far as the terms with large m, so that $a/(2m) \in (0, \frac{1}{2}\pi]$, are concerned. These are terms for which $m \ge a/\pi$. To that end,

$$\begin{aligned}
4&\left|\sum_{a/\pi \le m \le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| \\
&\le 4 \sum_{a/\pi \le m \le \log^2 N_1} \left|h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| \\
&= O\left(\sum_{a/\pi \le m \le \log^2 N_1} \frac{1}{N_1}\left(1 + \frac{4m^2}{a^2}\right)\right)
\end{aligned}$$

$$= O\left(\frac{1}{N_1} \sum_{a/\pi \le m \le \log^2 N_1} m^2\right) = O\left(\frac{\log^6 N_1}{N_1}\right).$$

Lastly, the sum

$$4 \sum_{1 \le m < a/\pi} \cos\left(\frac{a(2\theta - 1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right) \tag{2.4.14}$$

has a bounded number of terms. For each m, with $1 \le m < a/\pi$, we distinguish two cases. Either $a/(2m)$ is an integral multiple of π, or it is not. In the former case, we know that

$$h_{N_1+1,N_2}\left(\frac{a}{2m}\right) = 0,$$

and hence these terms do not have any contribution to the sum (2.4.14). For all the other values of m, with $1 \le m < a/\pi$, we examine the distances between the numbers $a/(2m\pi)$ and the set $\mathbb{Z}$. These distances, no matter how small, are some fixed strictly positive numbers, which are independent of N_1 and N_2. If we let $\delta > 0$ denote the smallest such distance, in other words,

$$\delta = \min\left\{\left\|\frac{a}{2\pi m}\right\| : 1 \le m < \frac{a}{\pi}, \quad \frac{a}{2m} \notin \mathbb{Z}\right\},$$

then

$$\begin{aligned} 4\left|\sum_{1 \le m < a/\pi} \cos\left(\frac{a(2\theta - 1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| &\le 4 \sum_{1 \le m < a/\pi} \left|h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| \\ &= O\left(\sum_{\substack{1 \le m < a/\pi \\ a/(2m) \notin \mathbb{Z}}} \frac{1}{N_1 \delta^2}\right) \\ &= O\left(\frac{1}{N_1 \delta^2}\right). \end{aligned}$$

Thus this sum too tends to 0 as $N_1 < N_2$ tend to infinity, since $\delta > 0$ is fixed.

In conclusion, for every $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N_1, N_2 > N(\epsilon)$,

$$|A(a, \theta, N_1) - A(a, \theta, N_2)| < \epsilon,$$

uniformly for all θ in any given compact subinterval of $(0, 1)$, as desired. Similarly, working with integrals instead of sums, we find that

$$|B(a, \theta, N_1) - B(a, \theta, N_2)| < \epsilon,$$

for N_1, N_2 sufficiently large. This implies that $S_5(a, \theta)$ is uniformly convergent on compact subsets of $(0, 1)$. The conclusion is that the initial sum $S(a, \theta)$ is uniformly convergent on compact subintervals of $(0, 1)$ if and only if $S_6(a)$ is. *But* $S_6(a)$ *does not depend on* θ*. So the convergence at one single value of* θ *implies uniform convergence in compact subintervals of* $(0, 1)$.

2.4.4 Reformulation and Proof of Entry 2.1.2

In view of Entry 2.1.2, Theorem 2.4.2, and the proof of convergence in Sect. 2.4.3, we now reformulate and prove the following theorem.

Theorem 2.4.3. *Fix $x > 0$ and set $\theta = u + \frac{1}{2}$, where $-\frac{1}{2} < u < \frac{1}{2}$. Recall that $F(x)$ is defined in (2.1.4). If the identity below is valid for at least one value of θ, then it is valid for all values of θ, and*

$$\sum_{1\leq n\leq x} (-1)^n F\left(\frac{x}{n}\right) \cos(2\pi n u) - \frac{1}{4} + x\log(2\cos(\pi u))$$
$$= \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}+u} \lim_{M\to\infty} \left\{ \sum_{m=1}^{\infty} \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right) dt \right\}$$
$$+ \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}-u} \lim_{M\to\infty} \left\{ \sum_{m=1}^{\infty} \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right) dt \right\}. \quad (2.4.15)$$

Moreover, the series on the right-hand side of (2.4.15) converges uniformly on compact subintervals of $(-\frac{1}{2}, \frac{1}{2})$.

Proof. For each nonnegative integer n, set

$$f_n(u) := \frac{1}{n+\frac{1}{2}+u} \lim_{M\to\infty} \left\{ \sum_{m=1}^{M} \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right) dt \right\}$$
$$+ \frac{1}{n+\frac{1}{2}-u} \lim_{M\to\infty} \left\{ \sum_{m=1}^{M} \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right) dt \right\}. \quad (2.4.16)$$

From our work in Sect. 2.4.3, we know that the series $\sum_{n=0}^{\infty} f_n(u)$ either diverges for each value of u or converges for each value of u with the convergence being uniform in every compact subinterval of $(-\frac{1}{2}, \frac{1}{2})$. Assuming that the latter holds, we define

$$f(u) := \sum_{n=0}^{\infty} f_n(u),$$

and we endeavor to prove that the two sides of (2.4.15) have the same Fourier coefficients. If $\tilde{f}(u)$ denotes the left-hand side of (2.4.15), then we want to show that

$$\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)e^{2\pi iku}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)e^{2\pi iku}du, \tag{2.4.17}$$

for each integer k. Since $\tilde{f}(u)$ as well as each of the functions $f_n(u)$, $n \geq 0$, is an even function of u, it is sufficient to show that for every integer $k \geq 0$,

$$\sum_{n=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)\cos(2\pi ku)du = 2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du. \tag{2.4.18}$$

In what follows, k is fixed, and we proceed under the aforementioned assumption of uniform convergence of the series $\sum_{n=0}^{\infty} f_n(u)$, so that the convergence at the left side of (2.4.18) is assured. Let us denote, for each positive integer N,

$$I_N := \sum_{n=0}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)\cos(2\pi ku)du,$$

so that (2.4.18) is equivalent to

$$\lim_{N\to\infty} I_N = 2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du. \tag{2.4.19}$$

Next, for N large, write I_N in the form

$$\begin{aligned} I_N = \sum_{n=0}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} &\frac{\cos(2\pi ku)}{n+\frac{1}{2}+u}\left(\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right)\right.\right.\\ &\left.-\int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right)dt\right\}\\ &+\frac{\cos(2\pi ku)}{n+\frac{1}{2}-u}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right)\right.\\ &\left.\left.-\int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right)dt\right\}\right)du. \end{aligned} \tag{2.4.20}$$

From Sect. 2.4.3, we know that for each fixed n, we have uniform convergence with respect to u on compact subintervals of $(-\frac{1}{2}, \frac{1}{2})$ as $M \to \infty$. Thus, in (2.4.20), we may interchange the order of summation, integration, and taking the limit as $M \to \infty$ to deduce that

$$I_N = \lim_{M\to\infty} \sum_{m=1}^{M} \sum_{n=0}^{N-1} \left\{ \int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12+u} \sin\left(\frac{2\pi(n+\frac12+u)x}{m}\right) du \right.$$
$$+ \int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12-u} \sin\left(\frac{2\pi(n+\frac12-u)x}{m}\right) du$$
$$- \int_0^M \int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12+u} \sin\left(\frac{2\pi(n+\frac12+u)x}{t}\right) du\, dt$$
$$\left. - \int_0^M \int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12-u} \sin\left(\frac{2\pi(n+\frac12-u)x}{t}\right) du\, dt \right\}. \tag{2.4.21}$$

For each n, $0 \le n \le N-1$, we rewrite the integrals with respect to u on the right side of (2.4.21) in the forms

$$\int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12+u} \sin\left(\frac{2\pi(n+\frac12+u)x}{m}\right) du$$
$$= \int_n^{n+1} \frac{\cos(2\pi k(w-n-\frac12))}{w} \sin\left(\frac{2\pi w x}{m}\right) dw$$
$$= (-1)^k \int_n^{n+1} \frac{\cos(2\pi k w)}{w} \sin\left(\frac{2\pi w x}{m}\right) dw$$

and

$$\int_{-\frac12}^{\frac12} \frac{\cos(2\pi k u)}{n+\frac12-u} \sin\left(\frac{2\pi(n+\frac12-u)x}{m}\right) du$$
$$= \int_n^{n+1} \frac{\cos(2\pi k(n+\frac12-w))}{w} \sin\left(\frac{2\pi w x}{m}\right) dw$$
$$= (-1)^k \int_{-n-1}^{-n} \frac{\cos(2\pi k w)}{w} \sin\left(\frac{2\pi w x}{m}\right) dw.$$

Similar calculations hold for the remaining two integrals in (2.4.21) with m replaced by t. Hence, (2.4.21) can be rewritten in the form

$$I_N = (-1)^k \lim_{M\to\infty} \left\{ \sum_{m=1}^{M} \int_{-N}^{N} \frac{\cos(2\pi k w)}{w} \sin\left(\frac{2\pi w x}{m}\right) dw \right.$$
$$\left. - \int_0^M \int_{-N}^{N} \frac{\cos(2\pi k w)}{w} \sin\left(\frac{2\pi w x}{t}\right) dw\, dt \right\}. \tag{2.4.22}$$

The first integral on the right side of (2.4.22) can be rewritten as

$$
\begin{aligned}
&\int_{-N}^{N} \frac{\cos(2\pi kw)}{w} \sin\left(\frac{2\pi wx}{m}\right) dw \\
&= \frac{1}{2}\int_{-N}^{N} \frac{\sin\left((2\pi k + 2\pi x/m)w\right)}{w} dw - \frac{1}{2}\int_{-N}^{N} \frac{\sin\left((2\pi k - 2\pi x/m)w\right)}{w} dw \\
&= \frac{1}{2}\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \frac{1}{2}\int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy.
\end{aligned}
$$

A similar representation holds for the last integral on the right-hand side of (2.4.22) with m replaced by t. Therefore, (2.4.22) can be recast in the form

$$
\begin{aligned}
I_N = \frac{(-1)^k}{2} \lim_{M\to\infty} \Bigg\{ &\sum_{m=1}^{M} \int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \\
&- \int_0^M \int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy\, dt + \int_0^M \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy\, dt \Bigg\}.
\end{aligned}
\tag{2.4.23}
$$

In the following we now need to assume that $k > 0$. For large m,

$$
\begin{aligned}
J_N(m) :&= \int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{m-1}^{m} \int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy\, dt \\
&= \int_{m-1}^{m} \left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy \right) dt \\
&= -\int_{m-1}^{m} \int_{(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy\, dt - \int_{m-1}^{m} \int_{-(2\pi k+2\pi x/t)N}^{-(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy\, dt.
\end{aligned}
\tag{2.4.24}
$$

Note that

$$
(2\pi k + 2\pi x/t)N \geq (2\pi k + 2\pi x/m)N \geq 2\pi kN,
$$

and so the integrand in each of the double integrals on the far right side of (2.4.24) is $O(1/N)$. Also, the two double integrals are over domains of area bounded by

$$
\frac{2\pi xN}{t} - \frac{2\pi xN}{m} = O\left(\frac{N}{mt}\right) = O\left(\frac{N}{m^2}\right).
$$

Hence, we see that the first double integral on the extreme right side of (2.4.24) is

$$
O\left(\frac{1}{m^2}\right).
$$

We now consider the second double integral on the far right side of (2.4.24). Note that

$$(2\pi k - 2\pi x/m)N \geq (2\pi k - 2\pi x/t)N \gg N.$$

Thus, it is easy to see that we will obtain the same estimates for the second double integral on the right-hand side of (2.4.24). We now sum both sides of (2.4.24), $[\log N]+1 \leq m \leq M$, to find that

$$\sum_{m=[\log N]+1}^{M} J_N(m) = O\left(\frac{1}{\log N}\right).$$

We now use the bound above in (2.4.23), so that (2.4.23) now reduces to

$$\begin{aligned} I_N = &\frac{(-1)^k}{2}\sum_{m=1}^{[\log N]}\left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N}\frac{\sin y}{y}dy\right)\\ &- \frac{(-1)^k}{2}\int_0^{[\log N]}\left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy\right)dt\\ &+ O\left(\frac{1}{\log N}\right). \end{aligned} \tag{2.4.25}$$

Next, we divide the sum on m into two parts, $m \leq \lceil 2x \rceil$ and $\lceil 2x \rceil < m \leq [\log N]$, and we similarly divide the interval of integration with respect to t. Note that for each $m \geq \lceil 2x \rceil + 1$ and every $t \in [m-1, m]$,

$$2\pi k - \frac{2\pi x}{m} \geq 2\pi k - \frac{2\pi x}{t} \geq 2\pi k - \frac{2\pi x}{\lceil 2x \rceil} \geq 2\pi k - \pi \geq \pi,$$

for all $k \geq 1$. Therefore, for such m, all the integrals in (2.4.25) are of the type, for $B \geq \pi N$,

$$\int_{-B}^{B}\frac{\sin y}{y}dy = \pi + O\left(\frac{1}{N}\right).$$

This estimate is uniform in m, for $m \geq \lceil 2x \rceil + 1$, and uniform in t, for $t \in [m-1, m]$. It follows that

$$\begin{aligned} &\int_{-(2\pi k\pm 2\pi x/m)N}^{(2\pi k\pm 2\pi x/m)N}\frac{\sin y}{y}dy - \int_{m-1}^{m}\int_{-(2\pi k\pm 2\pi x/t)N}^{(2\pi k\pm 2\pi x/t)N}\frac{\sin y}{y}dy\,dt\\ &\quad = \left(\pi + O\left(\frac{1}{N}\right)\right) - \left(\pi + O\left(\frac{1}{N}\right)\right) = O\left(\frac{1}{N}\right), \end{aligned}$$

uniformly for $m \geq \lceil 2x \rceil + 1$, where the $\pm$ signs above are the same in all four places, i.e., either all of the signs are plus, or all of the signs are minus. It follows that the ranges of summation and integration in (2.4.25) can be further reduced to a bounded range. Thus,

$$I_N = \frac{(-1)^k}{2} \sum_{m=1}^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \right)$$
$$- \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt$$
$$+ O\left(\frac{1}{\log N}\right). \tag{2.4.26}$$

Inside the sum on m, each integral has a limit as $N \to \infty$, and these limits are

$$\lim_{N\to\infty} \int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy = \pi, \qquad 1 \leq m \leq \lceil 2x \rceil,$$
$$\lim_{N\to\infty} \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy = \begin{cases} \pi, & \text{if } 2\pi k > 2\pi x/m, \\ 0, & \text{if } 2\pi k = 2\pi x/m, \\ -\pi, & \text{if } 2\pi k < 2\pi x/m. \end{cases}$$

In summary,

$$\lim_{N\to\infty} \frac{(-1)^k}{2} \sum_{m=1}^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \right)$$
$$= \frac{(-1)^k}{2} \left(\lceil 2x \rceil \pi - \# \{1 \leq m \leq \lceil 2x \rceil : m > x/k\} \pi \right.$$
$$\left. + \# \{1 \leq m \leq \lceil 2x \rceil : m < x/k\} \pi \right)$$
$$= \frac{(-1)^k \pi}{2} \left(\lceil 2x \rceil - \lceil 2x \rceil - \# \{1 \leq m \leq \lceil 2x \rceil : m = x/k\} \right.$$
$$\left. + 2\# \{1 \leq m \leq \lceil 2x \rceil : m \leq x/k\} \right)$$
$$= (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2} \delta, \tag{2.4.27}$$

where

$$\delta = \begin{cases} 1, & \text{if } x/k \text{ is an integer}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, by (2.4.26) and (2.4.27),

$$\lim_{N\to\infty} I_N = (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2} \delta$$
$$- \lim_{N\to\infty} \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt, \tag{2.4.28}$$

provided that the limit on the right-hand side of (2.4.28) indeed does exist. As we have seen above, the first integral on the right-hand side of (2.4.28) equals $\pi + O(1/N)$, uniformly in t, $t \in (0, \lceil 2x \rceil)$. Therefore,

$$\lim_{N\to\infty}\frac{(-1)^k}{2}\int_0^{\lceil 2x\rceil}\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy=\lim_{N\to\infty}\frac{(-1)^k}{2}\left(\lceil 2x\rceil\pi+O\left(\frac{1}{N}\right)\right)$$
$$=\frac{(-1)^k}{2}\lceil 2x\rceil\pi. \tag{2.4.29}$$

For the remaining double integral in (2.4.28), we subdivide the outer range of integration $[0,\lceil 2x\rceil]$ into the three ranges

$$\left[0,\frac{x}{k}-\frac{1}{\log N}\right],\qquad \left[\frac{x}{k}-\frac{1}{\log N},\frac{x}{k}+\frac{1}{\log N}\right],\qquad \left[\frac{x}{k}+\frac{1}{\log N},\lceil 2x\rceil\right].$$

Using the fact that

$$\sup_{B\in\mathbf{R}}\left|\int_{-B}^{B}\frac{\sin y}{y}dy\right|<\infty,$$

we find that

$$\int_{\frac{x}{k}-\frac{1}{\log N}}^{\frac{x}{k}+\frac{1}{\log N}}\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dydt=O\left(\frac{1}{\log N}\right). \tag{2.4.30}$$

Next, uniformly for $t\in\left[\frac{x}{k}+\frac{1}{\log N},\lceil 2x\rceil\right]$, we see that

$$\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy=\pi+O\left(\left(\left|2\pi k-\frac{2\pi x}{\frac{x}{k}+\frac{1}{\log N}}\right|N\right)^{-1}\right)$$
$$=\pi+O\left(\frac{\log N}{N}\right),$$

and hence

$$\int_{\frac{x}{k}+\frac{1}{\log N}}^{\lceil 2x\rceil}\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dydt$$
$$=\left(\lceil 2x\rceil-\left(\frac{x}{k}+\frac{1}{\log N}\right)\right)\left(\pi+O\left(\frac{\log N}{N}\right)\right)$$
$$=\lceil 2x\rceil\pi-\frac{\pi x}{k}+O\left(\frac{1}{\log N}\right). \tag{2.4.31}$$

Lastly, uniformly for $t\in\left(0,\frac{x}{k}-\frac{1}{\log N}\right)$,

$$\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy=-\pi+O\left(\left(\left|2\pi k-\frac{2\pi x}{\frac{x}{k}-\frac{1}{\log N}}\right|N\right)^{-1}\right)$$
$$=-\pi+O\left(\frac{\log N}{N}\right),$$

and hence

$$\int_0^{\frac{x}{k}-\frac{1}{\log N}} \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy dt = \left(\frac{x}{k} - \frac{1}{\log N}\right)\left(-\pi + O\left(\frac{\log N}{N}\right)\right)$$
$$= -\frac{\pi x}{k} + O\left(\frac{1}{\log N}\right). \tag{2.4.32}$$

Combining (2.4.29)–(2.4.32), we conclude that

$$\lim_{N\to\infty} \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt$$
$$= \frac{(-1)^k}{2}\left(\lceil 2x \rceil \pi - \lceil 2x \rceil \pi + \frac{\pi x}{k} + \frac{\pi x}{k}\right)$$
$$= \frac{(-1)^k \pi x}{k}. \tag{2.4.33}$$

Combining (2.4.33) and (2.4.28), we finally deduce that

$$\lim_{N\to\infty} I_N = (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2}\delta + \frac{(-1)^k \pi x}{k}. \tag{2.4.34}$$

So, assuming that the right-hand side of (2.4.15) converges for at least one value of θ, we see that either (2.4.15) or (2.4.19) is equivalent to the proposition that

$$(-1)^k \left[\frac{x}{k}\right] - \frac{(-1)^k}{2}\delta - \frac{(-1)^k x}{k} = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u) \cos(2\pi k u) du, \tag{2.4.35}$$

for each $k \geq 1$, where

$$\delta = \begin{cases} 1, & \text{if } x/k \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

There remains the calculation of the integral on the right-hand side of (2.4.35). First, for each $k \geq 1$,

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{1\leq n\leq x} (-1)^n F\left(\frac{x}{n}\right) \cos(2\pi n u)\cos(2\pi k u) du = (-1)^k F\left(\frac{x}{k}\right)$$
$$= (-1)^k \left(\left[\frac{x}{k}\right] - \frac{1}{2}\delta\right). \tag{2.4.36}$$

Trivially, for each $k \geq 1$,

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{1}{4}\cos(2\pi ku)du = 0. \tag{2.4.37}$$

Next, recall the Fourier series [126, p. 46, formula 1.441, no. 2]

$$\log(2\cos(\pi u)) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2\pi nu)}{n}, \qquad -\frac{1}{2} < u < \frac{1}{2}.$$

Because the series on the right-hand side above is boundedly convergent on $[-\frac{1}{2}, \frac{1}{2}]$, we may invert the order of summation and integration to deduce that

$$\begin{aligned} &2x\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(2\cos(\pi u))\cos(2\pi ku)du \\ &\quad = 2x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos(2\pi nu)\cos(2\pi ku)du \\ &\quad = x\frac{(-1)^{k-1}}{k}. \end{aligned} \tag{2.4.38}$$

Bringing together (2.4.36)–(2.4.38), we find that

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du = (-1)^k\left(\left[\frac{x}{k}\right] - \frac{1}{2}\delta\right) + x\frac{(-1)^{k-1}}{k}. \tag{2.4.39}$$

Comparing (2.4.39) with (2.4.35), we see that indeed (2.4.35) has been proven for $k \geq 1$.

Let us summarize what we have accomplished. We have assumed that (2.4.15) holds for one particular value of θ. We have shown that the right side of (2.4.15) converges uniformly on compact subsets of $(-\frac{1}{2}, \frac{1}{2})$. Thus, the right side is a well-defined, continuous function of θ on $(-\frac{1}{2}, \frac{1}{2})$, and we need to check that it is equal to the function on the left side of (2.4.15). Consider the difference of these two functions, which is a continuous function of θ on $(-\frac{1}{2}, \frac{1}{2})$. We have proved that all its Fourier coefficients for $k \neq 0$ vanish. Then, as a function of θ, this function will be constant. Moreover, since the two sides of (2.4.15) are equal for one particular value of θ, the aforementioned constant must be zero. And so (2.4.15) holds for all θ. This then completes the proof of Theorem 2.4.3. □

2.5 Proof of Ramanujan's Second Bessel Function Identity (Symmetric Form)

In this section, we prove Ramanujan's second assertion on page 335 of [269], i.e., Entry 2.1.2, under the assumption that the product of the indices of the double series tends to infinity. As in our proof of the first identity in symmetric form, it will be sufficient to prove Entry 2.1.2 for rational $\theta = a/q$, where q is prime and $0 < a < q$.

We define

$$\begin{aligned} &G(a,q,x)\\ &:= \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+a/q)x}\right)}{\sqrt{m(n+a/q)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-a/q)x}\right)}{\sqrt{m(n+1-a/q)}}\right\}\\ &= \frac{\sqrt{qx}}{2}\sum_{m=1}^{\infty}\sum_{\substack{r=0\\ r\equiv\pm a \bmod q}}^{\infty}\frac{I_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}}. \end{aligned} \tag{2.5.1}$$

Thus, Entry 2.1.2 is equivalent to the following theorem.

Theorem 2.5.1. *If q is prime and $0 < a < q$, then*

$$G(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right) - \frac{1}{4} + x\log(2\sin(\pi a/q)) =: K(a,q,x). \tag{2.5.2}$$

Our first task in reaching our goal of proving Entry 2.1.2 or Theorem 2.5.1 is to establish the following theorem.

Theorem 2.5.2. *If χ is a nonprincipal even primitive character modulo q, then*

$$\begin{aligned} \sum_{n\le x}{}' d_\chi(n) &= \frac{\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1\left(4\pi\sqrt{nx/q}\right)\\ &\quad - \frac{x}{\tau(\overline{\chi})}\sum_{h=1}^{q-1}\overline{\chi}(h)\log\left(2\sin(\pi h/q)\right). \end{aligned} \tag{2.5.3}$$

Proof. Recall the functional equation of $\zeta(2s)$ [101, p. 59],

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-(\frac{1}{2}-s)}\Gamma(\tfrac{1}{2}-s)\zeta(1-2s).$$

Recall also that if χ is an even nonprincipal primitive character of modulus q, then the Dirichlet L-function $L(x,\chi)$ satisfies the functional equation [101, p. 69]

$$(\pi/q)^{-s}\Gamma(s)L(2s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}(\pi/q)^{-(\frac{1}{2}-s)}\Gamma(\tfrac{1}{2}-s)L(1-2s,\overline{\chi}).$$

Then, if

$$F(s,\chi) := \zeta(2s)L(2s,\chi) = \sum_{n=1}^{\infty} d_\chi(n)n^{-2s}$$

and

$$\xi(s,\chi) := (\pi/\sqrt{q})^{-2s}\Gamma^2(s)F(s,\chi),$$

the functional equations of $\zeta(s)$ and $L(s,\chi)$ yield the functional equation

$$\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi\left(\tfrac{1}{2}-s,\overline{\chi}\right).$$

We next state a special case of [26, p. 351, Theorem 2; p. 356, Theorem 4]. In the notation of those theorems from [26], $q=0$, $r=\frac{1}{2}$, $m=2$, $\lambda_n = \mu_n = \pi^2 n^2/q$, $a(n) = d_\chi(n)$, and $b(n) = \tau(\chi)d_{\overline{\chi}}(n)/\sqrt{q}$. Also, as above, $J_\nu(x)$ denotes the ordinary Bessel function of order ν. Let $x>0$. Then

$$\sum_{\lambda_n\le x}{}' d_\chi(n) = \frac{\tau(\chi)}{\sqrt{q}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\left(\frac{x}{\mu_n}\right)^{1/4} K_{1/2}(4\sqrt{\mu_n x};-\tfrac{1}{2};2)+Q_0(x), \tag{2.5.4}$$

where [26, p. 348, Definition 4]

$$K_\nu(x;\mu;2) = \int_0^\infty u^{\nu-\mu-1}J_\mu(u)J_\nu(x/u)du$$

and

$$Q_0(x) = \frac{1}{2\pi i}\int_C \frac{(\pi/\sqrt{q})^{-2s}F(s,\chi)x^s}{s}ds,$$

where C is a positively oriented closed curve encircling the poles of the integrand. Moreover, the series on the right-hand side of (2.5.4) is uniformly convergent on compact intervals not containing values of λ_n.

We calculate $Q_0(x)$. Since $L(s,\chi)$ is an entire function, and since $L(0,\chi) = 0$, when the character χ is even, the only pole of the integrand is at $s=\frac{1}{2}$, arising from the simple pole of $\zeta(2s)$. Thus,

$$Q_0(x) = \frac{\sqrt{qx}}{\pi}L(1,\chi) = -\frac{\tau(\chi)}{\pi}\sqrt{\frac{x}{q}}\sum_{n=1}^{q-1}\overline{\chi}(n)\log|1-\zeta_q^n|, \tag{2.5.5}$$

where $\zeta_q = e^{2\pi i/q}$, and where we have used an evaluation for $L(1,\chi)$ found in [104].

Next, recall that [314, p. 54]

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \qquad \text{and} \qquad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z.$$

Thus, anticipating a later change of variable and using a result that can readily be derived from [314, p. 184, formula (3)], we find that

$$\begin{aligned}K_{1/2}(4\pi^2nx/q;-\tfrac{1}{2};2) &= \frac{1}{\pi^2}\sqrt{\frac{q}{nx}}\int_0^\infty \cos u \,\sin\left(\frac{4\pi^2nx}{qu}\right)du \\ &= -\frac{1}{\pi^2}\sqrt{\frac{q}{nx}}2\pi\sqrt{\frac{nx}{q}}\left(\frac{\pi}{2}Y_1(4\pi\sqrt{nx/q})+K_1(4\pi\sqrt{nx/q})\right) \\ &= I_1(4\pi\sqrt{nx/q}). \end{aligned} \tag{2.5.6}$$

We now replace x by π^2x^2/q and substitute the values $\lambda_n=\mu_n=\pi^2n^2/q$ in (2.5.4). Using (2.5.5) and (2.5.6) in (2.5.4), we conclude that

$$\sum_{n\le x}{}' d_\chi(n) = \frac{\tau(\chi)}{\sqrt{q}}\sum_{n=1}^\infty d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1(4\pi\sqrt{nx/q}) - \frac{\tau(\chi)x}{q}\sum_{n=1}^{q-1}\overline{\chi}(n)\log|1-\zeta_q^n|. \tag{2.5.7}$$

Using the fact that $\tau(\chi)\tau(\overline{\chi})=q$ and the simple identity

$$\log|1-\zeta_q^n| = \log|\zeta_q^{-n/2}-\zeta_q^{n/2}| = \log(2\sin(\pi n/q)),$$

we obtain

$$\begin{aligned}\sum_{n\le x}{}' d_\chi(n) = \frac{\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^\infty d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1\left(4\pi\sqrt{nx/q}\right) \\ - \frac{x}{\tau(\overline{\chi})}\sum_{n=1}^{q-1}\overline{\chi}(n)\log\left(2\sin(\pi n/q)\right),\end{aligned}$$

which completes the proof. □

We need one further result before commencing our proof of Theorem 2.5.1.

Lemma 2.5.1. *If* $0<a<q$ *and* $(a,q)=1$, *then*

$$\begin{aligned}&\sum_{n=1}^\infty F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right) \\ &= \sum_{1\le n\le x/q}{}' d(n) + \sum_{\substack{d|q\\d>1}}\frac{1}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi \text{ even}}}\chi(a)\tau(\overline{\chi})\sum_{1\le n\le dx/q}{}' d_\chi(n).\end{aligned}$$

The proof of Lemma 2.5.1 is very similar to that of Lemma 2.3.1, and so we omit the proof.

Proof of Theorem 2.5.1. First, using (2.3.10) and the fact that χ is even, we see that

$$G(a,q,x) = \frac{q}{2}\sum_{m=1}^\infty \sum_{\substack{r=1\\ r\equiv\pm a \bmod q}}^\infty \sqrt{\frac{x}{qmr}}I_1\left(4\pi\sqrt{mrx/q}\right)$$

$$
\begin{aligned}
&= \frac{q}{2\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sqrt{\frac{x}{qmr}} I_1\big(4\pi\sqrt{mrx/q}\big) \sum_{\chi \bmod q} \overline{\chi}(r)\big(\chi(a)+\chi(-a)\big) \\
&= \frac{q}{\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sqrt{\frac{x}{qmr}} I_1\big(4\pi\sqrt{mrx/q}\big) \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a)\overline{\chi}(r) \\
&= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \overline{\chi}(r) \sqrt{\frac{x}{qmr}} I_1\big(4\pi\sqrt{mrx/q}\big) \\
&= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\big(4\pi\sqrt{nx/q}\big).
\end{aligned}
$$

So, if q is prime and χ_0 denotes the principal character modulo q, then

$$
\begin{aligned}
G(a,q,x) &= \frac{q}{\phi(q)} \sum_{m=1}^{\infty} \sum_{\substack{r=1 \\ q\nmid r}}^{\infty} \sqrt{\frac{x}{qmr}} I_1\big(4\pi\sqrt{mrx/q}\big) \\
&\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\big(4\pi\sqrt{nx/q}\big) \\
&= \frac{q}{\phi(q)} \Delta(x/q) - \frac{1}{\phi(q)} \Delta(x) \\
&\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\big(4\pi\sqrt{nx/q}\big) \\
&= -\frac{1}{\phi(q)} {\sum_{n\le x}}' d(n) + \frac{q}{\phi(q)} {\sum_{n\le x/q}}' d(n) - \frac{1}{4} + \frac{x}{\phi(q)} \log q \\
&\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\big(4\pi\sqrt{nx/q}\big). \qquad (2.5.8)
\end{aligned}
$$

On the other hand, by Lemma 2.5.1 with q prime,

$$
\begin{aligned}
K(a,q,x) &= -\frac{1}{\phi(q)} {\sum_{n\le x}}' d(n) + \frac{1+\phi(q)}{\phi(q)} {\sum_{n\le x/q}}' d(n) - \frac{1}{4} \\
&\quad + \frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) {\sum_{1\le n\le x}}' d_{\chi}(n) + x\log(2\sin \pi a/q). \qquad (2.5.9)
\end{aligned}
$$

Thus, in view of (2.5.8), (2.5.9), and (2.5.2), it suffices to show that

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) {\sum_{1\leq n\leq x}}' d_{\chi}(n) + (q-1)x\log(2\sin\pi a/q)$$
$$= q \sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right) + x\log q.$$

By Theorem 2.5.2, we now only have to show that

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{h=1}^{q-1} \overline{\chi}(h) \log\left(2\sin(\pi h/q)\right) = (q-1)\log(2\sin\pi a/q) - \log q. \quad (2.5.10)$$

Now

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{h=1}^{q-1} \overline{\chi}(h) \log\left(2\sin(\pi h/q)\right) = \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right) \sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a)\overline{\chi}(h)$$
$$= \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right) \sum_{\chi \text{ even}} \chi(a)\overline{\chi}(h) - \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right)$$
$$= (q-1)\log(2\sin\pi a/q) - \log\left(2^{q-1}\prod_{h=1}^{q-1}\sin(\pi h/q)\right)$$
$$= (q-1)\log(2\sin\pi a/q) - \log q,$$

where we have used the familiar formula [126, p. 41, formula 1.392, no. 1]

$$\prod_{h=1}^{q-1}\sin(\pi h/q) = \frac{q}{2^{q-1}}.$$

Thus, (2.5.10) has been established, and we have completed the proof. □

3

Koshliakov's Formula and Guinand's Formula

3.1 Introduction

In his lecture at a conference to commemorate the centenary of Ramanujan's birth, held on June 1–5, 1987, at the University of Illinois at Urbana-Champaign, R. William Gosper remarked, "How can we pretend to love this man when he is forever reaching out from the grave to snatch away our neatest results?" In less colorful language, Gosper was asserting that it frequently happens that one proves an important theorem, only to discover later that it is ensconced somewhere in Ramanujan's writings. In other instances, we have learned that Ramanujan anticipated important later developments in his own inimitable way.

In this chapter, we examine two pages in Ramanujan's lost notebook [269, pp. 253–254], on one of which Gosper's observation is demonstrated once again. On page 253, Ramanujan states a version of a famous formula of A.P. Guinand, from which N.S. Koshliakov's equally famous formula follows as a corollary. On page 254, Ramanujan gives applications of Guinand's formula; these results are mostly new.

First, we discuss Koshliakov's formula. Koshliakov is chiefly remembered for one theorem, namely, *Koshliakov's formula* [188], which we now see was proved by Ramanujan about 10 years earlier. To state his formula, let $K_\nu(z)$ denote the modified Bessel function of order ν, defined in (2.1.3), and let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a > 0$,

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi an) = \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \quad (3.1.1)$$

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Koshliakov's proof, as well as most subsequent proofs, depends upon Voronoï's summation formula [310]

$$\begin{aligned}\sideset{}{'}\sum_{a\le n\le b} d(n)f(n) &= \int_a^b (\log x + 2\gamma) f(x)dx \\ &\quad + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\left(4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx})\right)dx, \quad (3.1.2)\end{aligned}$$

where $Y_\nu(z)$ denotes the Weber–Bessel function of order ν, defined in (2.1.2). The prime $\prime$ on the summation sign on the left-hand side indicates that if a or b is an integer, then only $\frac{1}{2}f(a)$ or $\frac{1}{2}f(b)$, respectively, is counted. For conditions on $f(x)$ that ensure the validity of (3.1.2), see, for example, Berndt's paper [28].

A.L. Dixon and W.L. Ferrar [112] also proved (3.1.1) using the Voronoï summation formula. F. Oberhettinger and K.L. Soni [235] established a generalization of (3.1.1) using Voronoï's formula (3.1.2), and she derived further identities from Koshliakov's formula [295]. In contrast to the work of these authors, Ramanujan evidently did not appeal to Voronoï's formula.

Koshliakov's formula can be considered an analogue of the transformation formula for the classical theta function, namely,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} = \sqrt{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau}, \quad \operatorname{Re}\tau > 0, \quad (3.1.3)$$

which, as is well known, is equivalent to the functional equation of the Riemann zeta function $\zeta(s)$ given by [306, p. 22]

$$\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1}{2}(1-s)\right)\zeta(1-s). \quad (3.1.4)$$

Ferrar [118] was evidently the first mathematician to prove indeed that (3.1.1) can be derived from the functional equation of $\zeta^2(s)$. Oberhettinger and Soni [235] showed that this functional equation and Koshliakov's formula are equivalent.

On page 253 in his lost notebook [269], Ramanujan states (3.1.1) as a corollary of a more general and especially beautiful formula at the top of the same page. This more general formula is stated in an equivalent formulation in Entry 3.1.1 below.

Entry 3.1.1 (p. 253). *Let* $\sigma_k(n) = \sum_{d|n} d^k$*, and let* $\zeta(s)$ *denote the Riemann zeta function. If* α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$*, and if* s *is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \tag{3.1.5}$$

The identity (3.1.5) is equivalent to a formula established by Guinand [136] in 1955. The series in Entry 3.1.1 are reminiscent of the Fourier expansion of nonanalytic Eisenstein series on $\mathbb{SL}(2,\mathbb{Z})$, or Maass wave forms [219], [226, pp. 230–232], [204, pp. 15–16], [304, pp. 208–209]. This Fourier series was published by H. Maass [219] in the language of Eisenstein series in the same year, 1949, that A. Selberg and S. Chowla [283], [282, pp. 367–378] published it in the similar vein of the Epstein zeta function, but with their proof not published until several years later [284], [282, pp. 521–545]. In the meanwhile, P.T. Bateman and E. Grosswald [24] published a proof. These Eisenstein series were shown by Maass [219] to satisfy a functional equation for automorphic forms. C.J. Moreno kindly informed the authors that he was easily able to derive Entry 3.1.1 from the aforementioned Fourier series expansion and functional equation. One may then regard (3.1.5) as an equivalent formulation of the functional equation for these nonholomorphic Eisenstein series or these particular Maass wave forms. The proof of Entry 3.1.1 that we give below is essentially the same as that of Guinand [136] and is completely independent of any considerations of nonanalytic Eisenstein series or their closely associated Epstein zeta functions. As is well known, Ramanujan made a large number of original contributions to Eisenstein series, many of which can be found in his lost notebook [13, Chaps. 11–16], [70].

On page 254, Ramanujan recorded formulas similar to Koshliakov's formula (3.1.1) or to Guinand's formula (3.1.5). We show that each of the three main results on this page can be deduced from Ramanujan's (and Guinand's) beautiful generalization (3.1.5) of Koshliakov's formula.

We close this introduction by mentioning two recent papers by S. Kanemitsu, Y. Tanigawa, H. Tsukada, and M. Yoshimoto [168] and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [171], in which the formulas of Koshliakov and Guinand are used or generalized.

The content of this chapter is taken from the second author's paper with Y. Lee and J. Sohn [62].

3.2 Preliminary Results

Throughout pages 253 and 254 of [269], Ramanujan expresses his theorems in terms of variants of the integral [126, p. 384, formula 3.471, no. 9]

$$\int_0^{\infty}x^{\nu-1}e^{-\beta/x-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_{\nu}(2\sqrt{\beta\gamma}), \tag{3.2.1}$$

where ν is any complex number and $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma > 0$. Since the modified Bessel function $K_\nu(z)$ is such a well-known function and its notation is standard, it seems advisable to avoid Ramanujan's notation for variants of (3.2.1), which he calls ϕ, ψ, and χ. In summary, we have converted all of Ramanujan's theorems to identities involving the modified Bessel function K_ν.

We use the well-known fact [126, p. 978, formula 8.469, no. 3]

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{3.2.2}$$

Necessary for us is the asymptotic behavior [314, p. 202]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \qquad z \to \infty,$$

which we invoke to ensure the convergence of series and integrals and also to justify the interchange of integration and summation several times in the sequel. We need several integrals of Bessel functions beginning with [126, p. 705, formula 6.544, no. 8]

$$\int_0^\infty K_\nu\left(\frac{a}{x}\right)K_\nu(bx)\frac{dx}{x^2} = \frac{\pi}{a}K_{2\nu}(2\sqrt{ab}), \quad \operatorname{Re}a > 0, \operatorname{Re}b > 0. \tag{3.2.3}$$

We need the related pair [295, p. 544, Eq. (8)]

$$\int_0^\infty xK_0(ax)K_0(bx)dx = \frac{\log(a/b)}{a^2-b^2}, \qquad a, b > 0, \tag{3.2.4}$$

and [126, p. 697, formula 6.521, no. 3]

$$\int_0^\infty xK_\nu(ax)K_\nu(bx)dx = \frac{\pi(ab)^{-\nu}(a^{2\nu}-b^{2\nu})}{2\sin(\pi\nu)(a^2-b^2)}, \qquad |\operatorname{Re}\nu| < 1,\ \operatorname{Re}(a+b) > 0. \tag{3.2.5}$$

Lastly, we need the evaluation [126, p. 708, formula 6.561, no. 16], for $\operatorname{Re}a > 0$ and $\operatorname{Re}(\mu+1\pm\nu) > 0$,

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{3.2.6}$$

3.3 Guinand's Formula

We begin by restating Entry 3.1.1.

Entry 3.3.1 (p. 253). *As usual, let* $\sigma_k(n) = \sum_{d|n} d^k$, *and let* $\zeta(s)$ *denote the Riemann zeta function. If* α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$, *and if* s *is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \tag{3.3.1}$$

To prove Entry 3.3.1, we need the following lemma.

Lemma 3.3.1. *Let $K_\nu(z)$ denote the modified Bessel function of order ν. If $x>0$ and $\operatorname{Re}\nu>0$, then*

$$\frac{1}{4}(\pi x)^{-\nu}\Gamma(\nu)+\sum_{n=1}^{\infty}n^{\nu}K_\nu(2\pi nx)$$
$$=\frac{1}{4}\sqrt{\pi}(\pi x)^{-\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)+\frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^{\infty}(n^2+x^2)^{-\nu-1/2}. \tag{3.3.2}$$

Lemma 3.3.1 is due to G.N. Watson [313], who proved it by using the Poisson summation formula. H. Kober [184] generalized Lemma 3.3.1 in two different directions. In one of them, the index n on the left-hand side of (3.3.2) was replaced by $n+\alpha$, $0<\alpha<1$, and in the other, $\cos(2\pi n\beta)$ was introduced into the summands on the left-hand side of (3.3.2). Berndt [32] generalized (3.3.2) by putting either an even or odd periodic sequence of coefficients in the infinite series of (3.3.2). The proof that we give below is essentially an elaboration of Guinand's proof [136].

Proof of Entry 3.3.1. Setting $n=kd$, we find that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)=\sqrt{\alpha}\sum_{n=1}^{\infty}\sum_{d|n}d^{-s}n^{s/2}K_{s/2}(2n\alpha)$$
$$=\sqrt{\alpha}\sum_{d=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k}{d}\right)^{s/2}K_{s/2}(2dk\alpha). \tag{3.3.3}$$

We now invoke Lemma 3.3.1 on the right-hand side above to deduce that for $\operatorname{Re}s>0$,

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)$$
$$=\sqrt{\alpha}\sum_{d=1}^{\infty}\frac{1}{d^{s/2}}\left(-\frac{1}{4}(d\alpha)^{-s/2}\Gamma\left(\frac{s}{2}\right)+\frac{1}{4}\sqrt{\pi}(d\alpha)^{-s/2-1}\Gamma\left(\frac{s+1}{2}\right)\right.$$
$$\left.+\frac{\pi^{3/2}}{2d\alpha}\left(\frac{d\alpha}{\pi^2}\right)^{s/2+1}\Gamma\left(\frac{s+1}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(n^2+(d\alpha/\pi)^2)^{(s+1)/2}}\right)$$

$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\alpha^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\alpha^2)^{(s+1)/2}} \tag{3.3.4}$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2/\pi^2+d^2)^{(s+1)/2}}$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2+d^2\pi^2)^{(s+1)/2}}, \tag{3.3.5}$$

where we used the hypothesis $\alpha\beta = \pi^2$. By symmetry, from (3.3.4), for $\operatorname{Re} s > 0$,

$$\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$= -\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\beta^2)^{(s+1)/2}}. \tag{3.3.6}$$

Reversing the roles of the summation variables d and n in (3.3.6), subtracting (3.3.6) from (3.3.5), and rearranging slightly, we deduce that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) - \frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1). \tag{3.3.7}$$

On the other hand, using the functional equation (3.1.4) of $\zeta(s)$ and the fact that $\alpha\beta = \pi^2$, we find that

$$\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1) = \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\pi^{s+1/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)$$
$$= \frac{1}{4}\alpha^{(-s-1)/2}(\alpha\beta)^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)$$
$$= \frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s). \tag{3.3.8}$$

Substituting (3.3.8) and its analogue with the roles of α and β reversed into (3.3.7), we find that

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)\\ =-\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\\ +\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)-\frac{1}{4}\alpha^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s).\end{aligned}\tag{3.3.9}$$

The identity (3.3.9) is simply a rearrangement of (3.3.1), and so the proof of (3.3.1) is complete for $\operatorname{Re} s>0$. By analytic continuation, (3.3.1) is valid for all complex numbers s. □

Since $K_s(z)=K_{-s}(z)$ [314, p. 79, Eq. (8)], we see that (3.1.5) is invariant under the replacement of s by $-s$.

Ramanujan completes page 253 with two corollaries, which we now state and prove.

Entry 3.3.2 (p. 253). *Let α and β be positive numbers such that $\alpha\beta=\pi^2$. Then*

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\alpha}-\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\beta}=\frac{\beta-\alpha}{12}+\frac{1}{4}\log\frac{\alpha}{\beta}.\tag{3.3.10}$$

Proof. Let $s=1$ in Entry 3.1.1. From (3.2.2),

$$\sqrt{\alpha n}K_{1/2}(2n\alpha)=\frac{1}{2}\sqrt{\pi}e^{-2n\alpha}.\tag{3.3.11}$$

Using (3.3.11), the values $\Gamma(-\frac{1}{2})=-2\Gamma(\frac{1}{2})=-2\sqrt{\pi}$ and $\zeta(-1)=-\frac{1}{12}$ [306, p. 19], and the Laurent expansion of $\zeta(s)$ about $s=1$ [306, p. 16, Eq. (2.1.16)] in (3.1.5), we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\alpha}-\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\beta}-\frac{\beta-\alpha}{12}\\ =\frac{1}{2\sqrt{\pi}}\lim_{s\to1}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}\\ =\frac{1}{2}\lim_{s\to1}\left(\frac{1}{s-1}+\gamma+\cdots\right)\\ \times\left(\left\{1-\frac{s-1}{2}\log\beta+\cdots\right\}-\left\{1-\frac{s-1}{2}\log\alpha+\cdots\right\}\right)\\ =\frac{1}{4}\log\frac{\alpha}{\beta}.\end{aligned}\tag{3.3.12}$$

We easily see that (3.3.12) is equivalent to (3.3.10), and so the proof is complete. □

Entry 3.3.2 is equivalent to the identity

$$\sum_{m=1}^{\infty}\frac{1}{m(e^{2m\alpha}-1)}-\sum_{m=1}^{\infty}\frac{1}{m(e^{2m\beta}-1)}=\frac{\beta-\alpha}{12}+\frac{1}{4}\log\frac{\alpha}{\beta}. \tag{3.3.13}$$

To see this, expand the summands in (3.3.13) in geometric series and collect all terms with the same exponents in the resulting double series. The formula (3.3.13) (or (3.3.10)) is equivalent to the transformation formula for the logarithm of the Dedekind eta function. Ramanujan stated (3.3.13) twice in his second notebook [268], namely as Corollary (ii) in Sect. 8 of Chap. 14 [38, p. 256] and as Entry 27(iii) in Chap. 16 [39, p. 43]. He also recorded (3.3.13) in an unpublished manuscript on infinite series reproduced with Ramanujan's lost notebook [269]; in particular, see formula (29) on page 320 of [269]. See also Chap. 12 in this volume or [42, p. 65, Entry 3.5].

We next demonstrate that Koshliakov's formula (3.1.1) is a corollary of Entry 3.3.1. Our proof is a detailed explication of that of Guinand [136].

Entry 3.3.3 (p. 253). *Let α and β denote positive numbers such that $\alpha\beta=\pi^2$. Then, if γ denotes Euler's constant,*

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\beta)+\sum_{n=1}^{\infty}d(n)K_0(2n\alpha)\right)\\=\sqrt{\beta}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\alpha)+\sum_{n=1}^{\infty}d(n)K_0(2n\beta)\right). \tag{3.3.14}$$

Proof. In order to let $s\to 0$ in Entry 3.1.1, we need the well-known Laurent expansions [126, p. 944, formula 8.321, no. 1]

$$\Gamma(s)=\frac{1}{s}-\gamma+\cdots \tag{3.3.15}$$

and [306, pp. 19–20, Eqs. (2.4.3) and (2.4.5)]

$$\zeta(s)=-\frac{1}{2}-\frac{1}{2}\log(2\pi)s+\cdots. \tag{3.3.16}$$

Hence, letting $s\to 0$ in (3.1.5) and using (3.3.15) and (3.3.16), we find that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}d(n)K_0(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}d(n)K_0(2n\beta) \tag{3.3.17}$$
$$=\frac{1}{4}\lim_{s\to 0}\left(\left\{\left(\frac{1}{s/2}-\gamma+\cdots\right)\left(-\frac{1}{2}-\frac{1}{2}\log(2\pi)s+\cdots\right)\right.\right.\\\left.\left.\times\left(\sqrt{\beta}\left\{1-\frac{1}{2}s\log\beta+\cdots\right\}-\sqrt{\alpha}\left\{1-\frac{1}{2}s\log\alpha+\cdots\right\}\right)\right\}\right.$$

$$+\left\{\left(\frac{1}{-s/2}-\gamma+\cdots\right)\left(-\frac{1}{2}+\frac{1}{2}\log(2\pi)s+\cdots\right)\right.$$
$$\left.\times\left(\sqrt{\beta}\left\{1+\frac{1}{2}s\log\beta+\cdots\right\}-\sqrt{\alpha}\left\{1+\frac{1}{2}s\log\alpha+\cdots\right\}\right)\right\}\Bigg)$$
$$=\frac{1}{4}\gamma(\sqrt{\beta}-\sqrt{\alpha})-\frac{1}{2}\log(2\pi)(\sqrt{\beta}-\sqrt{\alpha})+\frac{1}{4}(\sqrt{\beta}\log\beta-\sqrt{\alpha}\log\alpha)$$
$$=\frac{1}{4}\gamma(\sqrt{\beta}-\sqrt{\alpha})-\frac{1}{4}\log(4\alpha\beta)(\sqrt{\beta}-\sqrt{\alpha})+\frac{1}{4}(\sqrt{\beta}\log\beta-\sqrt{\alpha}\log\alpha),$$

where in the last step we used the equality $\alpha\beta=\pi^2$. A simplification and rearrangement of (3.3.17) yield (3.3.14) to complete the proof. □

3.4 Kindred Formulas on Page 254 of the Lost Notebook

Entry 3.4.1 (p. 254). *If* $a>0$,

$$\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = 2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an})$$
$$=\frac{a}{\pi^2}\sum_{n=1}^\infty \frac{d(n)\log(a/n)}{a^2-n^2}-\frac{1}{2}\gamma-\left(\frac{1}{4}+\frac{1}{4\pi^2 a}\right)\log a-\frac{\log(2\pi)}{2\pi^2 a}. \quad (3.4.1)$$

Proof. Expanding the integrand in geometric series, we find that

$$\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = \sum_{m=1}^\infty\sum_{k=1}^\infty \int_0^\infty \frac{1}{x}e^{-2\pi(mx+ak/x)}dx$$
$$=\sum_{m=1}^\infty\sum_{k=1}^\infty \int_0^\infty \frac{1}{u}e^{-2\pi(u+akm/u)}du$$
$$=\sum_{n=1}^\infty d(n)\int_0^\infty \frac{1}{u}e^{-2\pi(u+an/u)}du$$
$$=2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an}),$$

by (3.2.1), which proves the first part of (3.4.1).

The second identity in (3.4.1) was actually first proved in print in 1966 by Soni [295]. Her proof is short, depends on Koshliakov's formula (3.1.1), and uses the integral evaluations (3.2.3) with $\nu=0$ and (3.2.4). We use her idea to prove the second major claim of Ramanujan on page 254. □

In contrast to the claims on the top and bottom thirds of page 254, the one claim in the middle of page 254 seems to be missing one element, and so we shall proceed as we think Ramanujan might have done. Proceeding as we did above and employing (3.2.1), we find that

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= \sum_{m=1}^\infty\sum_{k=1}^\infty \frac{1}{\sqrt{m}}\int_0^\infty \frac{1}{\sqrt{u}}e^{-2\pi(u+akm/u)}du \\ &= \sum_{n=1}^\infty \sigma_{-1/2}(n)\int_0^\infty \frac{1}{\sqrt{u}}e^{-2\pi(u+an/u)}du \\ &= 2\sum_{n=1}^\infty \sigma_{-1/2}(n)(an)^{1/4}K_{1/2}(4\pi\sqrt{an}) \\ &= \frac{1}{\sqrt{2}}\sum_{n=1}^\infty \sigma_{-1/2}(n)e^{-4\pi\sqrt{an}}, \end{aligned} \tag{3.4.2}$$

where we have used (3.2.2). Ramanujan's next claim gives an identity for the last series above, with a replaced by $a/4$.

Entry 3.4.2 (p. 254). *For* $a > 0$,

$$\sum_{n=1}^\infty \sigma_{-1/2}(n)e^{-2\pi\sqrt{an}} = Ka\sum_{n=1}^\infty \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} + \text{two trivial terms.} \tag{3.4.3}$$

Evidently, K on the right-hand side of (3.4.3) represents an unspecified constant. Ramanujan does not divulge the identities of the "two trivial terms." Our calculation in (3.4.2), showing a discrepancy with the series on the left-hand side of (3.4.3), actually provides a clue that this series in (3.4.3) *should be replaced by the series on the right-hand side of* (3.4.2). We next state a corrected version of Entry 3.4.2 providing the identities of the constant and the "trivial terms."

Entry 3.4.3 (p. 254). *If* $a > 0$, *then*

$$\begin{aligned}\sum_{n=1}^\infty \sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} &- \frac{a}{\pi}\sum_{n=1}^\infty \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} \\ &= \frac{1}{2}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{\pi\sqrt{a}}-1\right)+\frac{1}{2}\zeta\left(-\frac{1}{2}\right)\left(4\pi\sqrt{a}-\frac{1}{\pi a}\right). \end{aligned} \tag{3.4.4}$$

Proof. In (3.1.5), set $s=\frac{1}{2}$ and $\alpha = x$, so that $\beta = \pi^2/x$. Then,

$$\begin{aligned}&\sqrt{x}\sum_{n=1}^\infty \sigma_{-1/2}(n)n^{1/4}K_{1/4}(2nx) - \frac{\pi}{\sqrt{x}}\sum_{n=1}^\infty \sigma_{-1/2}(n)n^{1/4}K_{1/4}(2n\pi^2/x) \\ &= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\frac{\sqrt{\pi}}{x^{1/4}}-x^{1/4}\right)+\frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\frac{\pi^{3/2}}{x^{3/4}}-x^{3/4}\right). \end{aligned} \tag{3.4.5}$$

Multiply both sides of (3.4.5) by

$$\frac{1}{x^{5/2}}K_{1/4}(2a\pi^2/x)$$

and integrate over $(0,\infty)$. Inverting the order of summation and integration by absolute convergence, we find that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^2}K_{1/4}(2nx)K_{1/4}(2a\pi^2/x)dx \\ &\quad-\pi\sum_{n=1}^{\infty}\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^3}K_{1/4}(2n\pi^2/x)K_{1/4}(2a\pi^2/x)dx \\ &=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\sqrt{\pi}I_3-I_1\right)+\frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\pi^{3/2}I_5-I_{-1}\right),\end{aligned} \tag{3.4.6}$$

where

$$I_j=\int_0^{\infty}u^{j/4}K_{1/4}(2a\pi^2u)du, \tag{3.4.7}$$

and where to obtain the four integrals on the right-hand side of (3.4.6), we made the change of variable $x=1/u$ in each one.

We examine each of the six integrals in (3.4.6) in turn. First, using (3.2.3) and (3.2.2), we find that

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^2}K_{1/4}(2nx)K_{1/4}(2a\pi^2/x)dx&=\frac{1}{2a\pi}K_{1/2}(4\pi\sqrt{an}) \\ &=\frac{1}{4\sqrt{2}a^{5/4}n^{1/4}\pi}e^{-4\pi\sqrt{an}}.\end{aligned} \tag{3.4.8}$$

Second, making the change of variable $u=\pi^2/x$ and using (3.2.5), we deduce that

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^3}K_{1/4}(2n\pi^2/x)K_{1/4}(2a\pi^2/x)dx&=\frac{1}{\pi^4}\int_0^{\infty}uK_{1/4}(2nu)K_{1/4}(2au)du \\ &=\frac{1}{\pi^4}\frac{\pi(4na)^{-1/4}(\sqrt{2n}-\sqrt{2a})}{2\sin(\pi/4)(4n^2-4a^2)} \\ &=\frac{\sqrt{2}(an)^{-1/4}}{8\pi^3(n+a)(\sqrt{n}+\sqrt{a})}.\end{aligned} \tag{3.4.9}$$

In our calculations of I_j, $j=3,1,5,-1$, we employ (3.2.6). Thus,

$$I_3=2^{-1/4}(2a\pi^2)^{-7/4}\Gamma(1)\Gamma\left(\frac{3}{4}\right)=\frac{1}{4a^{7/4}\pi^{7/2}}\Gamma\left(\frac{3}{4}\right), \tag{3.4.10}$$

$$I_1=2^{-3/4}(2a\pi^2)^{-5/4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)=\frac{1}{4a^{5/4}\pi^2}\Gamma\left(\frac{3}{4}\right), \tag{3.4.11}$$

$$I_5 = 2^{1/4}(2a\pi^2)^{-9/4}\Gamma\left(\frac{5}{4}\right)\Gamma(1) = \frac{1}{4a^{9/4}\pi^{9/2}}\Gamma\left(\frac{5}{4}\right), \tag{3.4.12}$$

$$I_{-1} = 2^{-5/4}(2a\pi^2)^{-3/4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{4a^{3/4}\pi}\Gamma\left(\frac{1}{4}\right). \tag{3.4.13}$$

Using (3.4.8)–(3.4.13) in (3.4.6) and making frequent use of the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we deduce that

$$\frac{1}{4\sqrt{2}a^{5/4}\pi}\sum_{n=1}^{\infty}\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} - \frac{1}{4\sqrt{2}a^{1/4}\pi^2}\sum_{n=1}^{\infty}\frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})}$$
$$= \frac{\sqrt{2}}{16}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{a^{7/4}\pi^2} - \frac{1}{a^{5/4}\pi}\right) + \frac{\sqrt{2}}{16}\zeta\left(-\frac{1}{2}\right)\left(-\frac{1}{a^{9/4}\pi^2} + \frac{4}{a^{3/4}}\right). \tag{3.4.14}$$

If we multiply both sides of (3.4.14) by $4\sqrt{2}a^{5/4}\pi$ and rearrange slightly, we obtain (3.4.4) to complete the proof. □

We record the last two results on page 254 as Ramanujan wrote them, except that we express the results in terms of Bessel functions. The constant K and the "two trivial terms" are not the same as they are in Entry 3.4.2.

Entry 3.4.4 (p. 254). *If* $a > 0$, *then*

$$\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = 2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}) \tag{3.4.15}$$

$$= Ka^2\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)} + \text{two trivial terms}. \tag{3.4.16}$$

Proof. We prove (3.4.15). Expanding the integrand in geometric series, setting $mx = u$, and invoking (3.2.1), we find that

$$\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m}\int_0^{\infty}e^{-2\pi(u+akm/u)}du$$
$$= \sum_{n=1}^{\infty}\sigma_{-1}(n)\int_0^{\infty}e^{-2\pi(u+an/u)}du$$
$$= 2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}).$$

□

Lastly, we provide and prove a more precise version of (3.4.16) giving the identities of the missing terms.

Entry 3.4.5 (p. 254). *If* $a > 0$ *and* γ *denotes Euler's constant, then*

$$2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})$$
$$= -\frac{a^2}{2\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)}+\frac{a}{2\pi}\left((\log a+\gamma)\zeta(2)+\zeta'(2)\right)+\frac{1}{4\pi}(\log 2a\pi+\gamma)+\frac{1}{48a\pi}. \tag{3.4.17}$$

Proof. In (3.3.10), set $\alpha = x$, so that $\beta = \pi^2/x$. Recalling (3.2.2), we find that

$$\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{nx}K_{1/2}(2nx)$$
$$= \left(\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\pi^2/x}-\frac{x}{12}\right)+\frac{1}{2}\log\frac{x}{\pi}+\frac{\pi^2}{12x}$$
$$=: I_1+I_2+I_3. \tag{3.4.18}$$

Next, multiply both sides of (3.4.18) by

$$\frac{1}{x^{5/2}}K_{1/2}(2a\pi^2/x)$$

and integrate over $(0,\infty)$.

Consider first the series arising on the left-hand side of (3.4.18). Inverting the order of summation and integration on the left-hand side by absolute convergence, we arrive at

$$\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx)K_{1/2}(2a\pi^2/x)dx$$
$$= \frac{1}{a\pi^{3/2}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}), \tag{3.4.19}$$

where we have employed (3.2.3).

Second, the contribution from I_3 in (3.4.18) is given by

$$\frac{\pi^2}{12}\int_0^{\infty}x^{-7/2}K_{1/2}(2a\pi^2/x)dx = \frac{\pi^2}{12}\int_0^{\infty}u^{3/2}K_{1/2}(2a\pi^2u)du = \frac{1}{96a^{5/2}\pi^{5/2}}, \tag{3.4.20}$$

where we used (3.2.6) in the last step with $\mu = \frac{3}{2}$, $\nu = \frac{1}{2}$, and a replaced by $2a\pi^2$.

Third, using (3.2.2), we find that the contribution from I_2 in (3.4.18) is equal to

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty x^{-5/2}\log(x/\pi)K_{1/2}(2a\pi^2/x)dx\\
&\quad=\frac{1}{4\sqrt{a\pi}}\int_0^\infty x^{-2}\log(x/\pi)e^{-2a\pi^2/x}dx\\
&\quad=\frac{1}{8a^{3/2}\pi^{5/2}}\int_0^\infty \log(2a\pi/u)e^{-u}du\\
&\quad=\frac{1}{8a^{3/2}\pi^{5/2}}\left\{\int_0^\infty e^{-u}\log(2a\pi)du-\int_0^\infty e^{-u}\log u\;du\right\}\\
&\quad=\frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi)-\int_0^\infty e^{-u}\log u\;du\right\}\\
&\quad=\frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi)+\gamma\right\}, \qquad (3.4.21)
\end{aligned}$$

since [126, p. 602, formula 4.331, no. 1]

$$\gamma=-\int_0^\infty e^{-u}\log u\;du.$$

Finally, the contribution from I_1 in (3.4.18) is given by

$$J:=\int_0^\infty\left(\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x}-\frac{1}{12}x\right)x^{-5/2}K_{1/2}(2a\pi^2/x)dx. \qquad (3.4.22)$$

Recall that $\zeta(2)=\pi^2/6$. Thus, we can write

$$\begin{aligned}
\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x}-\frac{1}{12}x&=\sum_{n=1}^\infty\sum_{d|n}\frac{1}{d}e^{-2n\pi^2/x}-\frac{1}{12}x\\
&=\sum_{d=1}^\infty\sum_{m=1}^\infty\frac{1}{d}e^{-2md\pi^2/x}-\frac{1}{12}x\\
&=\sum_{d=1}^\infty\frac{1}{d}\frac{1}{e^{2d\pi^2/x}-1}-\left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}. \qquad (3.4.23)
\end{aligned}$$

Using (3.4.23) and (3.2.2) in (3.4.22), we see that

$$\begin{aligned}
J&=\int_0^\infty\left(\sum_{n=1}^\infty\frac{1}{n}\frac{1}{e^{2n\pi^2/x}-1}-\left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}\right)\frac{1}{2\sqrt{a\pi}}e^{-2a\pi^2/x}\frac{dx}{x^2}\\
&=\frac{1}{2\sqrt{a\pi}}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n}\left(\frac{1}{e^{2n\pi^2/x}-1}-\frac{1}{2n\pi^2/x}\right)e^{-2a\pi^2/x}\frac{dx}{x^2}. \qquad (3.4.24)
\end{aligned}$$

Since for $z > 0$,

$$\frac{1}{e^z - 1} - \frac{1}{z} < 0,$$

we can change the order of summation and integration by the monotone convergence theorem. Hence,

$$\begin{aligned} J &= \frac{1}{2\sqrt{a\pi}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \left(\frac{1}{e^{2n\pi^2/x} - 1} - \frac{1}{2n\pi^2/x} \right) e^{-2a\pi^2/x} \frac{dx}{x^2} \\ &= \frac{1}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du. \qquad (3.4.25) \end{aligned}$$

Consider now two different expressions for the logarithmic derivative of the gamma function, namely [126, p. 952, formula 8.362, no. 1; formula 8.361, no. 8],

$$\begin{aligned} \frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \\ &= \log z - \frac{1}{z} - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-tz} dt, \end{aligned}$$

where $\operatorname{Re} z > 0$. Hence,

$$\int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du = \log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)}. \qquad (3.4.26)$$

Putting (3.4.26) in (3.4.25), we find that

$$\begin{aligned} J &= \frac{1}{4a^{1/2}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)} \right) \\ &= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) - \sum_{n=1}^{\infty} \frac{\log n}{n^2} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a}{n^2 m(mn+a)} \right) \\ &= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) + \zeta'(2) - a \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} \right) \\ &= -\frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \qquad (3.4.27) \end{aligned}$$

We now combine all our calculations that arose from (3.4.18), namely, (3.4.19)–(3.4.22), and (3.4.27), to deduce that

$$\begin{aligned} \frac{1}{a\pi^{3/2}} \sum_{n=1}^{\infty} \sigma_{-1}(n) \sqrt{n} K_1(4\pi\sqrt{an}) &= \frac{1}{96a^{5/2}\pi^{5/2}} + \frac{1}{8a^{3/2}\pi^{5/2}} \{\log(2a\pi) + \gamma\} \\ &\quad - \frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \qquad (3.4.28) \end{aligned}$$

Finally multiply both sides of (3.4.28) by $2\pi^{3/2}a^{3/2}$ to deduce (3.4.17) and complete the proof. □

Analogues of Guinand's formula in Entry 3.3.1 and Watson's lemma (Lemma 3.3.1) have been derived by Berndt [27]. These analogues are also discussed in the paper [62] on which this chapter is based. Analogues of Guinand's and Koshliakov's formulas with characters in the summands have been derived by Berndt, A. Dixit, and Sohn [52]. A different character analogue of Koshliakov's formula along with a connection to integrals of Dirichlet L-functions that are analogues of Ramanujan's famous integrals involving Riemann's Ξ-function [257] has been derived by Dixit [110]. H. Cohen [98] has continued the line of investigation represented by Entry 3.4.5 and has derived several interesting formulas of the same sort.

Dixit [107] has derived a beautiful extension of Koshliakov's formula. Recall that Riemann's ξ-function is defined by

$$\xi(s) := (s-1)\pi^{-s/2}\Gamma(1+\tfrac{1}{2}s)\zeta(s), \tag{3.4.29}$$

and that his Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it). \tag{3.4.30}$$

We now state Dixit's extension [107].

Theorem 3.4.1 (Extended version of Koshliakov's formula). *Let $\Xi(t)$ be defined by (3.4.30). If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}\sqrt{\alpha}&\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\alpha)\right)\\ &= \sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right)\\ &= -\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\tfrac{1}{2}t\right)\right)^2\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}.\end{aligned} \tag{3.4.31}$$

Dixit first showed that the far left side of (3.4.31) is equal to the integral on the far right-hand side. Next observe that if we put $\alpha = 1/\beta$ in this equality, then the first equality in (3.4.31) easily follows. Koshliakov [191] derived a formula similar to (3.4.31). Essentially, his formula arises from taking the Fourier cosine transform of both sides of (3.4.31).

Dixit [109] has also extended Guinand's formula.

Theorem 3.4.2 (Extended version of Guinand's formula). *If α and β are positive numbers such that $\alpha\beta = 1$, then for $-1 < \operatorname{Re} z < 1$,*

$$\begin{aligned}
&\sqrt{\alpha}\left(\alpha^{z/2-1}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-z/2-1}\pi^{z/2}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right.\\
&\qquad\left.-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}\left(2n\pi\alpha\right)\right)\\
&=\sqrt{\beta}\left(\beta^{z/2-1}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-z/2-1}\pi^{z/2}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right.\\
&\qquad\left.-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}\left(2n\pi\beta\right)\right)\\
&=-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt.
\end{aligned}\tag{3.4.32}$$

As with Dixit's extension of Koshliakov's formula, suppose that we can show that the far left side of (3.4.32) is equal to the far right side above. Then if we set $\alpha = 1/\beta$ in this equality, the first equality of (3.4.32) follows. Dixit [109] has obtained a companion theorem to Theorem 3.4.2 for $|\operatorname{Re} z| > 1$.

4

Theorems Featuring the Gamma Function

4.1 Introduction

In this chapter we collect scattered results from the lost notebook that involve the classical gamma function $\Gamma(z)$. In the next three sections, we consider the evaluations of three integrals involving the gamma function recorded on page 199 in [269]. Following these sections, we consider a very precise, fascinating approximation to the gamma function,

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \tag{4.1.1}$$

which is found on page 339 [269]. A slightly less precise forerunner appeared as a problem submitted by Ramanujan to the *Journal of the Indian Mathematical Society* [260], but a complete solution was never published in that journal. The inequalities (4.1.1) were proved by E.A. Karatsuba [177] in 2001 for $x \geq 1$ and by H. Alzer [4] in 2003 for $0 \leq x \leq 1$. In Sects. 4.5–4.8, we provide Karatsuba's elegant solution. Finally, in Sect. 4.9 we discuss a few miscellaneous claims.

4.2 Three Integrals on Page 199

In Chap. 13 of his second notebook [268], [38, pp. 226–227], Ramanujan briefly examined the problem of finding functions f such that

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{n=-\infty}^{\infty} f(n). \tag{4.2.1}$$

In particular, he incorrectly asserted that

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$$\int_{-\infty}^{\infty} \frac{a^x}{\Gamma(x+1)} dx = e^a. \tag{4.2.2}$$

If (4.2.2) were correct, then (4.2.2) would provide an example of (4.2.1), since $1/\Gamma(n+1) = 0$ when n is a negative integer. Authors examining instances of (4.2.1) include R.P. Boas and H. Pollard [72], P.J. Forrester [120], and K.S. Krishnan [202].

Page 199 in Ramanujan's lost notebook is devoted to three integral formulas, which can be considered attempts to give corrected versions of (4.2.2). Two of them are correct, but the remaining one is not, although it is true in certain cases.

Entry 4.2.1 (p. 199). *If $a > 0$ and $k \geq 0$, then*

$$\int_{-k}^{\infty} \frac{a^x}{\Gamma(x+1)} dx + \int_0^{\infty} \frac{e^{-ax} x^{k-1}}{\pi^2 + \log^2 x} \left(\cos \pi k - \frac{1}{\pi} \sin \pi k \log x \right) dx = e^a. \tag{4.2.3}$$

Entry 4.2.2 (p. 199). *If $a > 0$ and $k \geq 0$, then*

$$\int_{-k}^{\infty} \frac{a^x}{\Gamma(x+1)} dx + \frac{1}{2\pi} \int_0^{\infty} \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}} \Gamma(k+ix) + \frac{e^{-i\pi(k-ix)}}{a^{k-ix}} \Gamma(k-ix) \right\} dx = e^a. \tag{4.2.4}$$

Entry 4.2.3 (p. 199). *If $a > 0$, $0 \leq \lambda < \epsilon$, and $1/\epsilon$ is a positive integer, then*

$$\epsilon \sum_{n=0}^{\infty} \frac{a^{\lambda+n\epsilon}}{\Gamma(1+\lambda+n\epsilon)} = e^a - \frac{\epsilon}{\pi} \int_0^{\infty} e^{-ax} x^{-\lambda-1} \frac{\sin \pi(\lambda-\epsilon) - x^{\epsilon} \sin \pi\lambda}{2\cos \pi\epsilon - (x^{\epsilon} + x^{-\epsilon})} dx. \tag{4.2.5}$$

In the next section, we prove Entries 4.2.1 and 4.2.2, while in the subsequent section we discuss Entry 4.2.3. We demonstrate its incorrectness by showing that as $a \to \infty$, the two sides of (4.2.5) have different asymptotic expansions. On the other hand, if we regard the left side of (4.2.5) as a Riemann sum, then the limits of both sides as $\epsilon \to 0$ are equal.

The content of this portion of the chapter first appeared in a paper written by the second author [43].

4.3 Proofs of Entries 4.2.1 and 4.2.2

Proof of Entry 4.2.1. Let $f(a, k)$ denote the left side of (4.2.3). Then, by straightforward differentiation, the reflection formula for $\Gamma(k)$, and the standard integral representation for $\Gamma(k)$,

$$\begin{aligned}\frac{\partial}{\partial k}f(a,k) &= \frac{a^{-k}}{\Gamma(1-k)} + \int_0^\infty \frac{e^{-ax}x^{k-1}\log x}{\pi^2+\log^2 x}\left(\cos \pi k - \frac{1}{\pi}\sin \pi k \log x\right)dx \\ &\quad + \int_0^\infty \frac{e^{-ax}x^{k-1}}{\pi^2+\log^2 x}(-\pi \sin \pi k - \cos \pi k \log x)\,dx \\ &= \frac{a^{-k}}{\Gamma(1-k)} - \frac{\sin \pi k}{\pi}\int_0^\infty e^{-ax}x^{k-1}dx \\ &= \frac{a^{-k}}{\pi}\Gamma(k)\sin \pi k - \frac{\sin \pi k}{\pi}a^{-k}\Gamma(k) = 0. \end{aligned} \tag{4.3.1}$$

Hence, $f(a,k)$ is constant with respect to k.

Next, differentiating with respect to a and using the functional equation for the Γ-function, we find that

$$\begin{aligned}&\frac{\partial}{\partial a}f(a,k) \\ &\quad= \int_{-k}^\infty \frac{xa^{x-1}}{\Gamma(x+1)}dx - \int_0^\infty \frac{e^{-ax}x^k}{\pi^2+\log^2 x}\left(\cos \pi k - \frac{1}{\pi}\sin \pi k \log x\right)dx \\ &\quad= \int_{-k-1}^\infty \frac{a^x}{\Gamma(x+1)}dx + \int_0^\infty \frac{e^{-ax}x^k}{\pi^2+\log^2 x}\Bigg(\cos \pi(k+1) \\ &\qquad\qquad - \frac{1}{\pi}\sin \pi(k+1)\log x\Bigg)dx \\ &\quad= f(a,k+1). \end{aligned} \tag{4.3.2}$$

Thus, by (4.3.1) and (4.3.2),

$$\frac{\partial}{\partial a}f(a,k) = f(a,k).$$

It follows that for some constant c,

$$f(a,k) = ce^a. \tag{4.3.3}$$

It remains to evaluate c, and more precisely, in order to prove (4.2.3), we must show that $c=1$.

To prove that $c=1$, we evaluate $f(a,k)$ when $a=k=0$. From the definition of f and the substitution $u=\log x$, we see that

$$f(0,0) = \int_0^\infty \frac{dx}{x(\pi^2+\log^2 x)} = \int_{-\infty}^\infty \frac{du}{\pi^2+u^2} = 1. \tag{4.3.4}$$

Using (4.3.4) in (4.3.3), we conclude that $c=1$, as desired. □

Proof of Entry 4.2.2*.* Let $f(a,k)$ denote the left side of (4.2.4). Then

$$\begin{aligned}
&\frac{\partial}{\partial k} f(a,k)\\
&= \frac{a^{-k}}{\Gamma(1-k)} + \frac{i\pi}{2\pi}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) - \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx\\
&\quad - \frac{\log a}{2\pi}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) + \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx\\
&\quad + \frac{1}{2\pi}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\frac{\partial}{\partial k}\Gamma(k+ix) + \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\frac{\partial}{\partial k}\Gamma(k-ix)\right\} dx. \qquad (4.3.5)
\end{aligned}$$

Now by the chain rule,

$$\frac{\partial}{\partial k}\Gamma(k \pm ix) = \mp i \frac{\partial}{\partial x}\Gamma(k \pm ix). \qquad (4.3.6)$$

Hence, using (4.3.6) in (4.3.5) and integrating by parts, we find that

$$\begin{aligned}
&\frac{\partial}{\partial k} f(a,k)\\
&= \frac{a^{-k}}{\Gamma(1-k)} + \frac{i}{2}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) - \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx\\
&\quad - \frac{\log a}{2\pi}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) + \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx\\
&\quad + \frac{1}{2\pi i}\left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix)\Big|_0^\infty - \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\Big|_0^\infty \right\}\\
&\quad - \frac{1}{2\pi i}\int_0^\infty \left\{ \frac{-\pi e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) + \frac{\pi e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx\\
&\quad + \frac{i\log a}{2\pi i}\int_0^\infty \left\{ \frac{e^{i\pi(k+ix)}}{a^{k+ix}}\Gamma(k+ix) + \frac{e^{-i\pi(k-ix)}}{a^{k-ix}}\Gamma(k-ix)\right\} dx. \qquad (4.3.7)
\end{aligned}$$

Observe that by Stirling's formula for the Γ-function on a vertical line [126, p. 945], the expressions for the integrated terms vanish at ∞. After considerable cancellation, (4.3.7) reduces simply to

$$\begin{aligned}
\frac{\partial}{\partial k} f(a,k) &= \frac{a^{-k}}{\Gamma(1-k)} - \frac{1}{2\pi i}\left(\frac{e^{i\pi k}}{a^k} - \frac{e^{-i\pi k}}{a^k}\right)\Gamma(k)\\
&= \frac{a^{-k}}{\pi}\sin(\pi k)\Gamma(k) - \frac{a^{-k}}{\pi}\sin(\pi k)\Gamma(k) = 0, \qquad (4.3.8)
\end{aligned}$$

upon again using the reflection formula for the Γ-function. Thus, $f(a,k)$ is constant with respect to k.

Next, differentiating and using the functional equation of the Γ-function, we find that

$$\begin{aligned}\frac{\partial}{\partial a} f(a,k) &= \int_{-k-1}^{\infty} \frac{a^x}{\Gamma(x+1)} dx + \frac{1}{2\pi} \int_0^{\infty} \left\{ \frac{e^{i\pi(k+1+ix)}}{a^{k+1+ix}} \Gamma(k+1+ix) \right. \\ &\quad \left. + \frac{e^{-i\pi(k+1-ix)}}{a^{k+1-ix}} \Gamma(k+1-ix) \right\} dx \\ &= f(a,k+1). \end{aligned} \tag{4.3.9}$$

Hence, by (4.3.8) and (4.3.9),

$$\frac{\partial}{\partial a} f(a,k) = f(a,k),$$

and so, for some constant c,

$$f(a,k) = ce^a. \tag{4.3.10}$$

By (4.2.4), it remains to show that $c = 1$.

To determine c, we evaluate $f(a,k)$ when $a = 1$ and $k = 0$. Recalling that $f(a,k)$ denotes the left side of (4.2.4), we see that

$$f(1,0) = \int_0^{\infty} \frac{dx}{\Gamma(x+1)} + \frac{1}{2\pi} \int_0^{\infty} \left\{ e^{-\pi x} \left(\Gamma(ix) + \Gamma(-ix) \right) \right\} dx. \tag{4.3.11}$$

To evaluate the latter integral, examine, for $\epsilon > 0$,

$$I_\epsilon := \frac{1}{2\pi} \int_0^{\infty} \left\{ e^{-\pi x} \left(\Gamma(\epsilon + ix) + \Gamma(\epsilon - ix) \right) \right\} dx. \tag{4.3.12}$$

Inserting the integral representations for $\Gamma(\epsilon \pm ix)$ in (4.3.12) and inverting the order of integration by absolute convergence, we find that

$$\begin{aligned} I_\epsilon &= \frac{1}{2\pi} \left(\int_0^{\infty} \frac{e^{-t}}{t^{1-\epsilon}} dt \int_0^{\infty} e^{-x(\pi - i\log t)} dx + \int_0^{\infty} \frac{e^{-t}}{t^{1-\epsilon}} dt \int_0^{\infty} e^{-x(\pi + i \log t)} dx \right) \\ &= \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-t}}{t^{1-\epsilon}} \left(\frac{1}{\pi - i\log t} + \frac{1}{\pi + i \log t} \right) dt \\ &= \int_0^{\infty} \frac{e^{-t}}{t^{1-\epsilon}(\pi^2 + \log^2 t)} dt. \end{aligned} \tag{4.3.13}$$

Since

$$\int_0^{\infty} \frac{e^{-t}}{t(\pi^2 + \log^2 t)} dt < \infty,$$

by the Lebesgue dominated convergence theorem, we may take the limit as $\epsilon \to 0$ under the integral sign on the right side of (4.3.13). Thus, from (4.3.13) and (4.3.11),

$$f(1,0) = \int_0^{\infty} \frac{dt}{\Gamma(t+1)} + \int_0^{\infty} \frac{e^{-t}}{t(\pi^2 + \log^2 t)} dt = e, \tag{4.3.14}$$

by (4.2.3). Hence, by (4.3.10), $c = 1$, as desired. □

4.4 Discussion of Entry 4.2.3

We first observe that the sum on the left side of (4.2.5) is a Riemann sum in which each subinterval is of length ϵ, and the function $a^x/\Gamma(1+x)$ is evaluated at the point $\lambda+n\epsilon$ in the nth subinterval, where $n \geq 0$ and $0 \leq \lambda < \epsilon$. Thus, as $\epsilon \to 0$, the left side of (4.2.5) tends to the first integral on the left side of (4.2.3) when $k=0$. For the integral on the right side of (4.2.5), we can let $\epsilon \to 0$ inside the integral sign by the dominated convergence theorem. Recalling that $0 \leq \lambda < \epsilon$, we will assume that λ is a twice differentiable function of ϵ in applying L'Hospital's rule. After a straightforward, but not so short, calculation, we find that

$$\lim_{\epsilon\to 0} \frac{\epsilon(\sin \pi(\lambda-\epsilon) - x^{\epsilon}\sin \pi\lambda)}{2\cos \pi\epsilon - (x^{\epsilon}+x^{-\epsilon})} = \frac{\pi}{\pi^2+\log^2 x}.$$

Thus, letting $\epsilon \to 0$ on both sides of (4.2.5), we find that the proposed equality becomes

$$\int_0^{\infty} \frac{a^x}{\Gamma(x+1)}dx = e^a - \int_0^{\infty} \frac{e^{-ax}}{x(\pi^2+\log^2 x)}dx,$$

which is true, by (4.2.3) with $k=0$, and so Entry 4.2.3 is valid in the limit as $\epsilon \to 0$.

When $\lambda = 0$ and $\epsilon = 1$, then both sides of (4.2.5) are equal to e^a. Numerical calculations also show that for $\lambda = \frac{1}{2}$, $\epsilon = 1$, and small a, e.g., $a = 1, 2$, the two sides of (4.2.5) agree to at least 30 decimal places.

We now show that (4.2.5) is not valid in general. For simplicity, we choose $\lambda = \frac{1}{2}$ and $\epsilon = 1$ and show asymptotically that as $a \to \infty$, the left and right sides of (4.2.5) have different asymptotic expansions. Similar arguments are valid for other fixed values of λ and ϵ. Now

$$\frac{1}{\sqrt{x}(1+x)} = \sum_{n=0}^{\infty}(-1)^n x^{n-1/2}, \qquad 0<x<1,$$

and so by a routine application of Watson's Lemma [238, p. 113], the right side of (4.2.5) has the asymptotic expansion, as $a \to \infty$,

$$e^a - \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\frac{1}{2})}{a^{n+1/2}}. \tag{4.4.1}$$

On the left side of (4.2.5), we use a result of Ramanujan from Chap. 3 in his second notebook [268], [37, pp. 57, 58, Entry 10] along with the familiar asymptotic expansion [1, p. 257, formula 6.147]

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \tag{4.4.2}$$

as $x \to \infty$. A complete statement of Entry 10 in Chap. 3 is too long to give here, but it suffices to say that we are applying Entry 10 to $\phi(x) = \Gamma(x+1)/\Gamma(x+\frac{3}{2})$, which easily satisfies the theorem's hypotheses. Accordingly, we find from (4.4.2) that as $a \to \infty$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a^{n+1/2}}{\Gamma(n+\frac{3}{2})} &= \sqrt{a}\sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \\ &= e^a\sqrt{a}\left\{\frac{\Gamma(a+1)}{\Gamma(a+\frac{3}{2})} + O\left(\frac{1}{a^{5/2}}\right)\right\} \\ &= e^a\sqrt{a}\left\{a^{-1/2}\left(1-\frac{3}{8a}+O\left(\frac{1}{a^2}\right)\right)+O\left(\frac{1}{a^{5/2}}\right)\right\} \\ &= e^a\left\{1-\frac{3}{8a}+O\left(\frac{1}{a^2}\right)\right\}. \end{aligned} \tag{4.4.3}$$

A comparison of (4.4.3) with (4.4.1) shows that the left and right sides of (4.2.5) have different asymptotic expansions as $a \to \infty$, and so (4.2.5) cannot be true in general. In conclusion, however, we remark that in his notebooks, Ramanujan often wrote equality signs for asymptotic expansions and approximations; for example, he never used the symbols $\sim$ or $\approx$. Thus, it is most likely that Ramanujan himself did not regard (4.2.5) as an equality.

4.5 An Asymptotic Expansion of the Gamma Function

On page 339 in his lost notebook [269], Ramanujan states a remarkably interesting formula for the classical gamma function.

Entry 4.5.1 (p. 339). *If* $x \geq 0$, *then*

$$\Gamma(1+x) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{\theta_x}{30}\right)^{1/6}, \tag{4.5.1}$$

where θ_x *has the particular values*

$$\begin{aligned} \theta_0 &= \frac{30}{\pi^3} = 0.9675, \\ \theta_{1/12} &= 0.8071, \quad \theta_{7/12} = 0.3058, \\ \theta_{2/12} &= 0.6160, \quad \theta_{8/12} = 0.3014, \\ \theta_{3/12} &= 0.4867, \quad \theta_{9/12} = 0.3041, \\ \theta_{4/12} &= 0.4029, \quad \theta_{10/12} = 0.3118, \\ \theta_{5/12} &= 0.3509, \quad \theta_{11/12} = 0.3227, \\ \theta_{6/12} &= 0.3207, \quad \theta_1 = 0.3359, \\ \theta_{\infty} &= 1. \end{aligned}$$

Moreover,

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) \tag{4.5.2}$$
$$< \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}.$$

This entry is a more precise version of a problem that Ramanujan had submitted to the *Journal of the Indian Mathematical Society* [260], [267, p. 333], [65, p. 249], [49].

Question 754. *Show that*

$$e^x x^{-x}\pi^{-1/2}\Gamma(1+x) = (8x^3+4x^2+x+E)^{1/6},$$

where E lies between $\frac{1}{100}$ and $\frac{1}{30}$ for all positive values of x.

K.B. Madhava's partial solution in Volume 12 of the *Journal of the Indian Mathematical Society* does not yield the bounds for E proposed by Ramanujan. In Volume 13, E.H. Neville and C. Krishnamachary pointed out numerical errors in Madhava's solution, and consequently, Madhava's bounds for E are actually better than what he had originally claimed, but still not as sharp as those posed by Ramanujan. Neville and Krishnamachary conclude their remarks by writing, "Mr Ramanujam's assertion is seen to be credible, but more powerful means must be used if it is to be proved."

Before proving (4.5.2), we comment on the numerical values in the first portion of Entry 4.5.1. We checked each of the values of θ_x with Mathematica and found them to be correct, except for $\theta_{1/2}$ and $\theta_{11/12}$, for which the last recorded digit is incorrect (when rounded off). More precisely, $\theta_{1/2} = 0.320763$ and $\theta_{11/12} = 0.322766$.

In considering Ramanujan's claim, S. Ponnusamy and M. Vuorinen [240] defined the function

$$h(x) = (g(x))^6 - (8x^3+4x^2+x), \tag{4.5.3}$$

where

$$g(x) = \left(\frac{e}{x}\right)^x\frac{\Gamma(1+x)}{\sqrt{\pi}}, \tag{4.5.4}$$

and showed that in order to prove (4.5.2) for $x \geq 1$, it suffices to show that $h(x)$ is increasing from $(1,\infty)$ onto $(\frac{1}{100};\frac{1}{30})$. (See also the book [9, p. 476] by G. Anderson, M. Vamanamurthy, and M. Vuorinen.) (Observe that θ_x given in (4.5.1) is the same as the definition of $h(x)$ from (4.5.3).) Karatsuba [177] proved the conjecture about $h(x)$, and so the primary purpose in the following sections is to provide Karatsuba's proof and so in the process to prove Entry 4.5.1 for $x \geq 1$ as well. More precisely, we prove the following theorem of Karatsuba [177].

Theorem 4.5.1. *The function $h(x)$ is monotonically increasing from $(1,\infty)$ onto $(h(1),h(\infty))$ with*

$$h(1) = \frac{e^6}{\pi^3} - 13 = 0.0111976\ldots$$

and

$$h(\infty) = \frac{1}{30} = 0.0333\ldots.$$

Before proving Theorem 4.5.1, we offer some remarks on further work by H. Alzer [4] and C. Mortici [227, 228]. As remarked earlier, Alzer proved that (4.5.2) also holds for $x \in (0,1)$. However, in contrast to $[1,\infty)$, where Karatsuba proved that $h(x)$ is strictly increasing, $h(x)$ is not monotonic on $(0,1)$. Computer calculations indicate that $h(x)$ has a local maximum and a local minimum on $(0,1)$. More precisely, it appears that $h(x)$ is increasing on $[0,a]$, decreasing on $[a,b]$, and increasing on $[b,1]$, where

$$a \approx 0.007714449 \qquad \text{and} \qquad b \approx 0.671503766.$$

Moreover,

$$h(a) \approx 0.033250349.$$

This is very interesting. Since $h(\infty) = \frac{1}{30} = 0.0333\ldots$, we find that the value at the local maximum obtained at $x = a$ is almost equal to the absolute maximum of $h(x)$ obtained at ∞. From Ramanujan's calculations of $\theta_x = 30h(x)$ in Entry 4.5.1, it is not clear whether Ramanujan realized that $h(x)$ is increasing near $x = 0$ and then shortly thereafter achieves a local maximum at $x = a$. If he had noticed this, he likely would have highlighted this behavior with a few values of θ_x for x between 0 and $\frac{1}{12}$. However, Ramanujan's calculations do indicate that $h(x)$ achieves a local minimum between $\frac{2}{3}$ and $\frac{3}{4}$, and it is quite likely that he saw that $\frac{2}{3}$ is very close to the point at which this local minimum is achieved. Alzer also showed that Ramanujan's lower bound of $\frac{1}{100}$ for $h(x)$ can be replaced by

$$\min_{0.6 \le x \le 0.7} h(x) = 0.010045071\ldots,$$

and that this is the best possible result. Karatsuba had previously shown that the upper bound of $\frac{1}{30}$ is indeed best possible. We will not prove Alzer's theorem here. Mortici [227–229] further improved the work of Alzer and Karatsuba by proving the following sharper inequality. For $x \ge 8$,

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} - \frac{11}{240x}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} - \frac{10}{240x}\right)^{1/6}. \quad (4.5.5)$$

The leftmost inequality in (4.5.5) actually holds for $x \ge 2$. An improvement on Mortici's theorem for positive integral values of the argument was achieved by Hirschhorn [161**?**], who proved that

$$\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1}{30}-\frac{11}{240n}+\frac{5}{240n^2}\right)^{1/6}$$
$$<\Gamma(n+1)<\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1}{30}-\frac{11}{240n}+\frac{9}{240n^2}\right)^{1/6}.$$

Yet another asymptotic formula for the gamma function has been derived by G. Nemes [231], who compares his formula with that of Ramanujan.

To prove Theorem 4.5.1, three lemmas are necessary.

Lemma 4.5.1. *For $x \geq x_0 = 2.4$, the function $h(x)$ satisfies the inequalities*

$$\frac{1}{100} < h(x) < \frac{1}{30}.$$

Moreover, $h(x) \to \frac{1}{30}$, as $x \to \infty$.

Lemma 4.5.2. *For $x \geq x_1 = 4.21$, the function $h(x)$ is monotonically increasing.*

Lemma 4.5.3. *For $1 < x \leq \max(x_0, x_1) = 4.21$, the function $h(x)$ is monotonically increasing.*

Our proofs naturally rely on the asymptotic formulas of Stirling for the functions $\log \Gamma(x)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$, as well as computer calculations for $1 < x \leq 4.21$. For the latter, we shall not give details but refer to the work of Karatsuba [177].

4.6 An Integral Arising in Stirling's Formula

We first write $h(x)$ in terms of an integral arising from Stirling's formula for $\log \Gamma(x)$. To that end, take logarithms of both sides of (4.5.4) to deduce that

$$\log g(x) = x - x\log x + \log x - \log\sqrt{\pi} + \log\Gamma(x). \tag{4.6.1}$$

Now recall Stirling's formula for $\log \Gamma(x)$ in the form [176, pp. 342–343]

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \log\sqrt{2\pi} + J(x), \tag{4.6.2}$$

where

$$J(x) := \int_0^\infty \frac{\sigma(u)du}{(x+u)^2}, \tag{4.6.3}$$

$$\sigma(u) := \int_0^u \rho(t)dt, \tag{4.6.4}$$

and

$$\rho(t) = \frac{1}{2} - \{t\}, \tag{4.6.5}$$

and where $\{t\}$ denotes the fractional part of t. Substitute (4.6.2) into (4.6.1) to deduce that

$$\log g(x) = \log \sqrt{2x} + J(x),$$

or

$$g(x) = \sqrt{2x} e^{J(x)}.$$

From this and from (4.5.3), we deduce the useful representation

$$h(x) = 8x^3 e^{6J(x)} - (8x^3 + 4x^2 + x). \tag{4.6.6}$$

We next determine an asymptotic expansion for $J(x)$. From (4.6.4) and (4.6.5), we easily see that

$$\sigma(u + 1) = \sigma(u)$$

and

$$\sigma(u) = \sigma(\{u\}) = \int_0^{\{u\}} \left(\frac{1}{2} - t \right) dt = \frac{1}{2}\{u\}(1 - \{u\}).$$

Hence, for $0 \leq u \leq 1$,

$$\sigma(u) = \frac{1}{2}u(1 - u), \quad \sigma(0) = \sigma(1) = 0, \quad \sigma'(u) = \frac{1}{2} - u.$$

Now expand $\sigma(u)$ in a Fourier series

$$\sigma(u) = \sum_{n=-\infty}^{\infty} c(n) e^{2\pi i n u}, \tag{4.6.7}$$

where the coefficients $c(n)$ are given by

$$c(0) = \int_0^1 \sigma(u) du = \int_0^1 \frac{1}{2} u(1 - u) du = \frac{1}{12}, \tag{4.6.8}$$

$$c(n) = \int_0^1 \sigma(u) e^{-2\pi i n u} du = \frac{1}{4\pi^2 n^2}, \quad n \neq 0, \tag{4.6.9}$$

upon two integrations by parts. Hence, from (4.6.9), (4.6.7), and (4.6.8),

$$\sigma(u) = \frac{1}{12} - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{4\pi^2 n^2} e^{2\pi i n u} = \frac{1}{12} - \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2} \cos(2\pi n u). \tag{4.6.10}$$

Substituting the last expression in (4.6.10) into (4.6.3), we deduce that

$$J(x) = \frac{1}{12}\int_0^\infty \frac{du}{(u+x)^2} - \frac{1}{2\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \frac{\cos(2\pi nu)du}{(u+x)^2}$$
$$= \frac{1}{12x} - \frac{1}{2\pi^2}\sum_{n=1}^\infty \frac{J_0(x;n)}{n^2},$$

where

$$J_0(x;n) := \int_0^\infty \frac{\cos(2\pi nu)du}{(u+x)^2}.$$

Recall that the Bernoulli numbers B_n, $n \geq 0$, can be defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}, \qquad |x| < 2\pi.$$

Recall also that $B_{2n+1} = 0$, $n \geq 1$, and that $\operatorname{sgn} B_{2n} = (-1)^{n-1}$, $n \geq 1$. Using this last fact about Bernoulli numbers and integrating $J_0(x;n)$ by parts, we arrive at the well-known classical asymptotic formula

$$J(x) = \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} + R_n(x), \tag{4.6.11}$$

where, since $R_n(x) > 0$ if n is odd and $R_n(x) < 0$ if n is even,

$$|R_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)x^{2n-1}}. \tag{4.6.12}$$

In particular, setting $n = 3$ in (4.6.11) and (4.6.12), we deduce that

$$J(x) = \frac{1}{12x} - \frac{1}{360x^3} + R_3(x), \tag{4.6.13}$$

where

$$0 < R_3(x) < \frac{1}{1260x^5}. \tag{4.6.14}$$

4.7 An Asymptotic Formula for $h(x)$

From the last two expressions, (4.6.13) and (4.6.14), we can represent $e^{6J(x)}$ in the form

$$e^{6J(x)} = e^{1/(2x)}e^{-\alpha}, \tag{4.7.1}$$

where

$$\alpha = \alpha(x) = \frac{1}{60x^3} - R, \tag{4.7.2}$$

and where

$$R = R(x), \quad 0 < R < \frac{1}{210x^5}. \tag{4.7.3}$$

By (4.7.2) and (4.7.3), we easily see that for $x \geq 1$,

$$0 < \alpha \leq \frac{1}{60x^3}.$$

From the Maclaurin expansion of $e^{-\alpha}$, $\alpha > 0$, we obtain the simple inequalities

$$1 - \alpha \leq e^{-\alpha} \leq 1 - \alpha + \frac{\alpha^2}{2},$$

and so, for $x > 0$ sufficiently large,

$$1 - \frac{1}{60x^3} + R \leq e^{-\alpha} \leq 1 - \frac{1}{60x^3} + R + \frac{1}{2}\left(\frac{1}{60x^3} - R\right)^2,$$

from which we can deduce that

$$1 - \frac{1}{60x^3} \leq \ e^{-\alpha} \leq 1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6}.$$

From the last inequality and (4.7.1), we arrive at

$$e^{1/(2x)}\left(1 - \frac{1}{60x^3}\right) \leq e^{6J(x)} \leq e^{1/(2x)}\left(1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6}\right).$$

From the inequalities above and from the Taylor series for $e^{1/(2x)}$ about $x = \infty$, we can deduce the bounds

$$e^{6J(x)} \geq \left(1 - \frac{1}{60x^3}\right)\left(1 + \frac{1}{2x} + \frac{1}{2!(2x)^2} + \frac{1}{3!(2x)^3} + \frac{1}{4!(2x)^4} + \frac{1}{5!(2x)^5}\right), \tag{4.7.4}$$

$$e^{6J(x)} \leq \left(1 - \frac{1}{60x^3} + \frac{1}{210x^5} + \frac{9}{39200x^6}\right)\left(1 + \frac{1}{2x} + \frac{1}{2!(2x)^2} + \cdots\right). \tag{4.7.5}$$

We first derive a lower bound for $h(x)$. From (4.7.4),

$$\begin{aligned} h(x) &= 8x^3e^{6J(x)} - (8x^3 + 4x^2 + x) \\ &\geq \left(8x^3 + 4x^2 + x + \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2}\right)\left(1 - \frac{1}{60x^3}\right) - (8x^3 + 4x^2 + x) \\ &\geq \frac{1}{30} - \frac{11}{240x} - \frac{7}{480x^2} - \frac{1}{360x^3} - \frac{1}{2880x^4} - \frac{1}{28800x^5}. \end{aligned} \tag{4.7.6}$$

We can now conclude from (4.7.6) that for $x \geq 2.4$,

$$h(x) > 0.0114 > h(1). \tag{4.7.7}$$

Second, we obtain an upper bound for $h(x)$ from (4.7.5). Set

$$S(x) := 1 - \frac{1}{60x^3} + \frac{1}{210x^5}, \tag{4.7.8}$$

$$T(x) := 1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{48x^3} + \frac{1}{384x^4} + \frac{1}{3840x^5}, \tag{4.7.9}$$

$$\delta(x) := \frac{9}{39200x^6}T(x) + \left(S(x) + \frac{9}{39200x^6}\right)\left(\frac{1}{6!(2x)^6} + \frac{1}{7!(2x)^7} + \cdots\right), \tag{4.7.10}$$

so we can rewrite (4.7.5) in the form

$$e^{6J(x)} \le S(x)T(x) + \delta(x). \tag{4.7.11}$$

Observe that for $x \ge 1$,

$$\begin{aligned}\frac{1}{6!(2x)^6} + \frac{1}{7!(2x)^7} + \cdots &\le \frac{1}{6!(2x)^6}\left(1 + \frac{1}{7}\left(\frac{1}{(2x)} + \frac{1}{(2x)^2} + \cdots\right)\right) \\ &\le \frac{1}{6!(2x)^6}\frac{3}{2} = \frac{1}{30720x^6},\end{aligned} \tag{4.7.12}$$

and also that for $x \ge 1$,

$$S(x) \le 1, \quad T(x) \le \frac{33}{20}.$$

Thus, from (4.7.10) and (4.7.9),

$$0 \le \delta(x) \le \frac{297}{784000x^6} + \left(1 + \frac{9}{39200x^6}\right)\frac{1}{30720x^6} \le \frac{21}{50000x^6}. \tag{4.7.13}$$

From (4.6.6), (4.7.11), and (4.7.13), we then conclude that

$$h(x) \le 8x^3S(x)T(x) - (8x^3 + 4x^2 + x) + \delta_1, \tag{4.7.14}$$

where

$$0 \le \delta_1 = 8x^3\delta(x) \le \frac{21}{6250x^3}. \tag{4.7.15}$$

Next, from (4.7.8)–(4.7.10), (4.7.14), and (4.7.15),

$$\begin{aligned}&8x^3S(x)T(x) - (8x^3 + 4x^2 + x) \\ &= \left(8x^3 + 4x^2 + x + \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2}\right)\left(1 - \frac{1}{60x^3} + \frac{1}{210x^5}\right) \\ &\quad - (8x^3 + 4x^2 + x) \\ &= \frac{1}{6} + \frac{1}{48x} + \frac{1}{480x^2} - \frac{8}{60} - \frac{4}{60x} - \frac{1}{60x^2} - \frac{1}{360x^3} - \frac{1}{2880x^4} - \frac{1}{28800x^5} \\ &\quad + \frac{4}{105x^2} + \frac{2}{105x^3} + \frac{1}{210x^4} + \frac{1}{1260x^5} + \frac{1}{10080x^6} + \frac{1}{100800x^7} \\ &= \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \delta_2(x),\end{aligned} \tag{4.7.16}$$

where

$$0 \leq \delta_2(x) \leq \frac{41}{2520x^3} + \frac{89}{20160x^4} + \frac{17}{22400x^5} + \frac{1}{10080x^6} + \frac{1}{100800x^7} \leq \frac{1}{46x^3}. \tag{4.7.17}$$

Hence, from (4.7.14), (4.7.16), and (4.7.17), we obtain the upper bound

$$h(x) \leq \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{21}{6250x^3} + \frac{1}{46x^3} \leq \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{251}{10000x^3}.$$

Rewriting the last inequality in the form

$$h(x) \leq \frac{1}{30} - \left(\frac{11}{240x} - \frac{79}{3360x^2} - \frac{251}{10000x^3} \right), \tag{4.7.18}$$

we can easily test that for $x \geq 1.04$, the value of the expression within parentheses on the right side of (4.7.18) is a positive number. Consequently,

$$h(x) \leq \frac{1}{30}, \qquad \text{for } x \geq 1.04. \tag{4.7.19}$$

Moreover, from (4.7.6) and (4.7.18), we see that

$$h(x) \to \frac{1}{30}, \qquad \text{as } x \to \infty. \tag{4.7.20}$$

From (4.7.7), (4.7.19), and (4.7.20), we obtain the assertion of Lemma 4.5.1.

4.8 The Monotonicity of $h(x)$

Differentiating (4.5.3) and (4.5.4), we find that, respectively,

$$h'(x) = 6g'(x)g^5(x) - (24x^2 + 8x + 1) \tag{4.8.1}$$

and

$$g'(x) = g(x) \left(\frac{1}{x} + \psi(x) - \log x \right). \tag{4.8.2}$$

Differentiating Stirling's formula (4.6.2) and (4.6.3), we arrive at

$$\psi(x) = \log x - \frac{1}{2x} + J'(x),$$

where

$$J'(x) = -2 \int_0^\infty \frac{\sigma(u)du}{(x+u)^3}. \tag{4.8.3}$$

Hence, we can rewrite (4.8.1) and (4.8.2) in their respective forms

$$g'(x) = g(x)\left(\frac{1}{2x} + J'(x)\right),$$
$$h'(x) = 48x^3 e^{6J(x)}\left(\frac{1}{2x} + J'(x)\right) - (24x^2 + 8x + 1). \tag{4.8.4}$$

To prove Lemma 4.5.2, it will be necessary for us to prove that $h'(x) > 0$, for $x \geq x_1$, or that from (4.8.4),

$$\frac{1}{2x} + J'(x) > \left(\frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3}\right) e^{-6J(x)}. \tag{4.8.5}$$

From (4.6.13) and (4.6.14),

$$-6J(x) = -\frac{1}{2x} + \frac{1}{60x^3} - \frac{\eta_1}{210x^5}, \tag{4.8.6}$$

where $0 \leq \eta_1 \leq 1$. Inserting (4.8.6) into (4.8.5), we find that

$$\frac{1}{2x} + J'(x) > \left(\frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3}\right) e^{-1/(2x)} e^{1/(60x^3)} e^{-\eta_1/(210x^5)}, \tag{4.8.7}$$

where $0 \leq \eta_1 \leq 1$. To prove the inequality (4.8.7), it suffices to prove it for $\eta_1 = 0$. Hence, we prove that

$$e^{-1/(60x^3)}\left(\frac{1}{2x} + J'(x)\right) > \left(\frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3}\right) e^{-1/(2x)}. \tag{4.8.8}$$

Substitute the Taylor expansions around the origin for the exponential functions in (4.8.8) and diminish the left-hand side and augment the right-hand side of (4.8.8), using the relations

$$1 - \beta \leq e^{-\beta} \leq 1 - \beta + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \frac{\beta^4}{4!}.$$

Hence, in order to prove (4.8.8), it suffices to prove that

$$\left(1 - \frac{1}{60x^3}\right)\left(\frac{1}{2x} + J'(x)\right)$$
$$> \left(\frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3}\right)\left(1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{1}{384x^4}\right). \tag{4.8.9}$$

Using an argument similar to that used in deriving the asymptotic expansion (4.6.11) of $J(x)$, we can deduce that

$$J'(x) = -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{\eta_2}{120x^8}, \tag{4.8.10}$$

where $0 \le \eta_2 \le 1$. It now follows that

$$J'(x) > -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}.$$

Hence, in order to establish (4.8.9), it suffices to prove the inequality

$$\left(1 - \frac{1}{60x^3}\right)\left(\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}\right) > \left(\frac{1}{2x} + \frac{1}{6x^2} + \frac{1}{48x^3}\right)\left(1 - \frac{1}{2x} + \frac{1}{8x^2} - \frac{1}{48x^3} + \frac{1}{384x^4}\right). \tag{4.8.11}$$

Multiplying the expressions in parentheses in (4.8.11), we see that (4.8.11) reduces to the inequality

$$-\frac{1}{252x^6} + \frac{1}{720x^5} - \frac{1}{7200x^7} + \frac{1}{15120x^9} > \frac{1}{2304x^5} + \frac{1}{18432x^7},$$

or

$$\frac{11}{11520} > \frac{1}{252x} + \frac{89}{460800x^2} - \frac{1}{15120x^4}.$$

This last inequality holds when $x \ge 4.21$. Therefore, with $x \ge x_1 = 4.21$, we also deduce that the inequality (4.8.5) holds. From this and from (4.8.4), we draw the conclusion that for $x \ge x_1 = 4.21$, the function $h(x)$ is monotonically increasing. This completes the proof of Lemma 4.5.2.

Since by (4.8.1) and (4.8.2),

$$h'(1) = 6\frac{e^6}{\pi^3}(1-\gamma) - 33 = 0.00558319\ldots > 0, \tag{4.8.12}$$

it follows from (4.8.12) that to prove Lemma 4.5.3, it will suffice to prove that

$$h'(x) > 0 \tag{4.8.13}$$

for each x, $1 < x \le \max(x_0, x_1) = x_1 = 4.21$. To accomplish this, we use the mean value theorem in the form

$$h'(x+d) - h'(x) = dh''(x + \vartheta d) \tag{4.8.14}$$

for some ϑ such that $0 \le \vartheta \le 1$. Karatsuba [177] applies (4.8.14) in intervals of length $d = 0.0001$. In order to show that (4.8.13) holds, bounds for the derivatives $h''(x)$ and $J''(x)$ similar to those obtained for $h'(x)$ and $J'(x)$ must be obtained. Since the analysis is similar to the previous analysis, we ask readers to consult Karatsuba's clear and detailed exposition for the remaining details of the proof of Lemma 4.5.3.

Theorem 4.5.1 easily follows from Lemmas 4.5.1–4.5.3. Except for the values of θ_x claimed by Ramanujan, Entry 4.5.1 is a direct consequence of Theorem 4.5.1.

Before closing this section, we comment on a related theorem of Alzer [5]. R. Windschitl had noted that for $x > 8$, the approximation

$$\Gamma(x+1) \approxeq \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2}$$

gives at least eight decimal places of the gamma function. This inspired Alzer to prove the following theorem [5].

Theorem 4.8.1. *For all* $x > 0$*,*

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right),$$

where the best possible constants are $\alpha = 0$ *and* $\beta = 1/1620$*.*

In a personal communication [6] to the second author, Alzer offered comments comparing the bounds in his Theorem 4.8.1 with those in Ramanujan's Entry 4.5.1. To that end, define

$$R(x) := \left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6}, \quad A(x) := \sqrt{2x}\left(x\sinh\frac{1}{x}\right)^{x/2},$$
$$S(x) := \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad B(x) := \sqrt{2x}\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right).$$

With the use of MAPLE V, he showed that

$$\begin{aligned} A(x) - R(x) &< 0, \qquad \text{for } x \in (0,1),\\ S(x) - B(x) &< 0, \qquad \text{for } x \in (0, 0.99), \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to\infty} x^{5/2}\left(A(x)-R(x)\right) &= \frac{7\sqrt{2}}{14400},\\ \lim_{x\to\infty} x^{7/2}\left(S(x)-B(x)\right) &= \frac{11\sqrt{2}}{11520}. \end{aligned}$$

Hence, for $x \in [0, 0.99]$, Ramanujan's upper and lower bounds for $\Gamma(x+1)$ are superior to the bounds given in Theorem 4.8.1, whereas for large x, the opposite is true. The fact that Ramanujan provided detailed calculations of θ_x for $x \in (0,1)$ indicates that he also thought that the primary interest for his inequalities was in this interval.

4.9 Pages 214, 215

Pages 214 and 215 contain scratch work that is very difficult to decipher. Except for one result, all decipherable claims can be found in Ramanujan's published papers.

Entry 4.9.1 (p. 214). *For* $0 < a < b - \frac{1}{2}$*,*

$$\int_0^\infty \left(\frac{1 + x^2/b^2}{1 + x^2/a^2}\right)\left(\frac{1 + x^2/(b+1)^2}{1 + x^2/(a+1)^2}\right)\left(\frac{1 + x^2/(b+2)^2}{1 + x^2/(a+2)^2}\right)\cdots dx$$
$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b-a-\frac{1}{2})}{\Gamma(a)\Gamma(b-\frac{1}{2})\Gamma(b-a)}. \quad (4.9.1)$$

We have quoted one of Ramanujan's formulas from [255], [267, p. 54, Eq. (3)]. To obtain the formula on page 214, replace b by $a+1$ and a by $b+1$ in (4.9.1).

Entry 4.9.2 (p. 214). *For* $a, b > 0$*,*

$$\int_0^\infty \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\cdots\{1 + x^2/b^2\}\{1 + x^2/(b+1)^2\}\cdots}$$
$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\frac{1}{2})}. \quad (4.9.2)$$

The identity (4.9.2) is Ramanujan's formula (17) from [255], [267, p. 57]. To obtain the result on page 214, replace a by $a+1$ and b by $b+1$ in (4.9.2).

Entry 4.9.3 (p. 214). *If either* $\mathrm{Re}(a+b) > \frac{3}{2}$ *or* $2(a-b)$ *is an odd integer and* $\mathrm{Re}(a+b) > 1$*, then*

$$\int_0^\infty \frac{dx}{\Gamma(a+x)\Gamma(a-x)\Gamma(b+x)\Gamma(b-x)}$$
$$= \frac{\Gamma(2a+2b-3)}{2\Gamma(2a-1)\Gamma(2b-1)\{\Gamma(a+b-1)\}^2}. \quad (4.9.3)$$

We have exactly recorded Ramanujan's evaluation on page 214, which is the same as in his paper [266], [267, p. 226, Eq. (7.12)], except that in his paper [266], a is replaced by α and b is replaced by β.

The next result is recorded on both pages 214 and 215, except that on page 215, the integrand is expressed in terms of product representations of gamma functions. The result can be found as Eq. (1.22) in Ramanujan's paper [266], [267, p. 216].

Entry 4.9.4 (pp. 214, 215). *If* $\operatorname{Re} a > \frac{1}{2}$, *then*

$$\int_0^\infty \frac{dx}{\Gamma(a+x)\Gamma(a-x)} = \frac{2^{2a-3}}{\Gamma(2a-1)}.$$

The other result on page 215 that we are able to read is not connected with the gamma function, but there is no other logical place to put it.

Entry 4.9.5 (p. 215). *For* $n > 0$,

$$\int_0^\infty \frac{\cos(nx)}{1-x^2}dx = \frac{\pi}{2}\sin n. \tag{4.9.4}$$

Of course, the integral in (4.9.4) must be interpreted as a principal value. It is interesting that for a more general result [126, p. 446, formula 3.723, no. 9], namely,

$$\int_0^\infty \frac{\cos(ax)}{b^2-x^2}dx = \frac{\pi}{2b}\sin(ab), \qquad a, b > 0, \tag{4.9.5}$$

the editors of [126] also fail to indicate that the integral in question diverges and should be replaced by a principal value. Let $a = n$ and $b = 1$ in (4.9.5) to obtain (4.9.4).

5

Hypergeometric Series

5.1 Introduction

The purpose of this chapter is to discuss two entries on page 200 and two on page 327 in Ramanujan's lost notebook. All four entries fall under the purview of hypergeometric series. We begin with the two entries on page 200.

On page 200 of his lost notebook, Ramanujan offers two results on certain bilateral hypergeometric series. As we shall see, the second follows from a theorem of J. Dougall [113]. The first gives a formula for the derivative of a quotient of two particular bilateral hypergeometric series. Ramanujan's formula needs to be slightly corrected, but what is remarkable is that such a formula exists! This is one of those instances in which we can undauntedly claim that if Ramanujan had not discovered the formula, no one else, at least in the foreseeable future, would have done so. Our proofs of these two formulas first appeared in a paper by the second author and W. Chu [50].

We first state the second formula, which requires modest deciphering, because of Ramanujan's use of ellipses to denote missing terms. It will be used in the proof of Ramanujan's first formula on page 200.

Entry 5.1.1 (p. 200). *Let α, β, γ, δ, and ξ be complex numbers such that* $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>3$. *Then*

$$\begin{aligned}\sum_{n=-\infty}^{\infty} & \frac{\xi+2n}{\Gamma(\alpha+\xi+n)\Gamma(\beta-\xi-n)\Gamma(\gamma+\xi+n)\Gamma(\delta-\xi-n)\Gamma(\alpha-n)} \\ & \times \frac{1}{\Gamma(\beta+n)\Gamma(\gamma-n)\Gamma(\delta+n)} \\ = & \frac{\sin(\pi\xi)\,\Gamma(\alpha+\beta+\gamma+\delta-3)}{\pi\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)} \\ & \times \frac{1}{\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}. \end{aligned} \tag{5.1.1}$$

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We secondly state a corrected version of Ramanujan's more interesting formula, i.e., the first formula. At the end of Sect. 5.4, we indicate the mistakes in Ramanujan's original formula.

Entry 5.1.2 (Corrected, p. 200). *Define, for real numbers* s *and* θ, $0<\theta<2\pi$, *and for any complex numbers* α, β, γ, *and* δ *such that* $\operatorname{Re}(\alpha+\beta+\gamma+\delta)>4$,

$$\varphi_s(\theta) := \sum_{n=-\infty}^{\infty} \frac{e^{(n+s)i\theta}}{\Gamma(\alpha+s+n)\Gamma(\beta-s-n)\Gamma(\gamma+s+n)\Gamma(\delta-s-n)}. \tag{5.1.2}$$

Then

$$\frac{d}{d\theta}\frac{\varphi_s(\theta)}{\varphi_t(\theta)} = \frac{i\sin\{\pi(s-t)\}\left(2\sin\frac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}}{\pi\varphi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} \tag{5.1.3}$$

On page 327 in his lost notebook [269], Ramanujan offers two beautiful continued fractions connected with hypergeometric polynomials, which we now offer.

Entry 5.1.3 (p. 327). *Let*

$$\varphi(a,x) := \frac{1}{\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+3}\right)^2\right\}\left\{1+\left(\frac{x}{a+5}\right)^2\right\}\cdots}. \tag{5.1.4}$$

Then, for $a+1>0$, $b+1>0$, *and* s *not purely imaginary,*

$$\int_0^\infty \varphi(a,x)\varphi(b,x)\frac{dx}{1+s^2x^2} = 2\sqrt{\pi}\,\frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+\frac{b}{2}\right)\Gamma\left(1+\frac{a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)} \times \frac{1}{a+b+1}\,{+}\,\frac{1(a+1)(b+1)(a+b+1)s^2}{a+b+3} \,{+}\,\frac{2(a+2)(b+2)(a+b+2)s^2}{a+b+5}\,{+}\cdots.$$

Entry 5.1.4 (p. 327). *If* $s=1$, *the continued fraction in Entry 5.1.3 can be written in the form*

$$\frac{1}{a+b+1}\,{+}\,\frac{1(a+1)(b+1)(a+b+1)}{a+b+3}\,{+}\,\frac{2(a+2)(b+2)(a+b+2)}{a+b+5}\,{+}\cdots = \frac{1}{a+b+1}(1-A_1+A_1A_2-A_1A_2A_3+\cdots), \tag{5.1.5}$$

where

$$A_t = \frac{(a+t)(b+t) - ab\cos^2\dfrac{\pi t}{2}}{(a+1+t)(b+1+t) - ab\cos^2\dfrac{\pi t}{2}}. \tag{5.1.6}$$

If we set $\alpha = (a+1)/2$ and $\beta = (b+1)/2$ and replace x with $2x$, then Entry 5.1.3 can be recast in the following form.

Entry 5.1.5 (p. 327). *Let*

$$\phi(\alpha, x) := \frac{1}{\left\{1+\left(\dfrac{x}{\alpha}\right)^2\right\}\left\{1+\left(\dfrac{x}{\alpha+1}\right)^2\right\}\left\{1+\left(\dfrac{x}{\alpha+2}\right)^2\right\}\cdots}. \tag{5.1.7}$$

Then, for $\alpha > 0$, $\beta > 0$, and s not purely imaginary,

$$\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\frac{dx}{1+4s^2x^2} = \sqrt{\pi}\frac{\Gamma\left(\alpha+\frac12\right)\Gamma\left(\beta+\frac12\right)\Gamma\left(\alpha+\beta\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\alpha+\beta+\frac12\right)}\chi_1(s),$$

where

$$\chi_1(s) := \frac{1}{2+}\,\frac{2\cdot 1(2\alpha)(2\beta)s^2}{2\alpha+2\beta+1+}\,\frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{2\alpha+2\beta+3} \\ +\,\frac{3(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)s^2}{2\alpha+2\beta+5}+\cdots. \tag{5.1.8}$$

These continued fractions are connected with the continuous Hahn polynomials. In his Ph.D. thesis [318], J. Wilson found a remarkably general class of orthogonal hypergeometric polynomials, in which all of the classical and several additional polynomials can be expressed as special or limiting cases. In particular, certain ${}_3F_2$ polynomials with two free parameters, called the continuous symmetric Hahn polynomials, were found by R. Askey and Wilson [16]. They are defined for all nonnegative integers n by

$$P_n(x) := P_n(x;\alpha,\beta) := i^n\,{}_3F_2\left(\begin{matrix}-n, n+2\alpha+2\beta-1, \beta-ix \\ \alpha+\beta, 2\beta\end{matrix};1\right) \tag{5.1.9}$$

and are orthogonal with respect to the positive absolutely continuous weight function

$$W(x) := |\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2, \tag{5.1.10}$$

where $-\infty < x < \infty$ and $\alpha, \beta > 0$ or $\alpha = \bar{\beta}$ and $\operatorname{Re}\alpha > 0$.

In Sects. 5.7 and 5.8, we provide two entirely different proofs of Entry 5.1.3, and in Sect. 5.9, we prove Entry 5.1.4. These proofs are due to S.-Y. Kang, S.-G. Lim, and J. Sohn [175]. The first proof of Entry 5.1.3 is instructive, because it relates Ramanujan's result to Hahn polynomials and the moment problem. The second proof is undoubtedly closer to Ramanujan's approach than the first, because it relies in the beginning stages on a theorem in Ramanujan's paper [255].

5.2 Background on Bilateral Series

For every integer n, define

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{5.2.1}$$

The bilateral hypergeometric series ${}_pH_p$ is defined for complex parameters $a_1, a_2, \dots, a_p$ and $b_1, b_2, \dots, b_p$ by

$${}_pH_p \left[\begin{matrix} a_1, a_2, \dots, a_p; \\ b_1, b_2, \dots, b_p; \end{matrix} z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_p)_n} z^n.$$

With the use of D'Alembert's ratio test, it can be checked that ${}_pH_p$ converges only for $|z| = 1$, provided that [290, p. 181, Eq. (6.1.1.6)]

$$\operatorname{Re}(b_1 + b_2 + \cdots + b_p - a_1 - a_2 - \cdots - a_p) > 1. \tag{5.2.2}$$

The series ${}_pH_p$ is said to be well-poised if

$$a_1 + b_1 = a_2 + b_2 = \cdots = a_p + b_p.$$

In 1907, Dougall [113] proved that a well-poised series ${}_5H_5$ can be evaluated at $z = 1$. In order to state this evaluation, define

$$\Gamma \left[\begin{matrix} a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_n \end{matrix}\right] := \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m)}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_n)}.$$

Then Dougall's formula [290, p. 182, Eq. (6.1.2.5)] is given by

$$\begin{aligned} &{}_5H_5 \left[\begin{matrix} 1+\frac{1}{2}a, & b, & c, & d, & e; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e; \end{matrix} 1\right] \\ &\quad = \Gamma \left[\begin{matrix} 1-b, 1-c, 1-d, 1-e, 1+a-b, 1+a-c, 1+a-d, \\ 1+a, 1-a, 1+a-b-c, 1+a-b-d, 1+a-b-e, \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} 1+a-e, 1+2a-b-c-d-e \\ 1+a-c-d, 1+a-c-e, 1+a-d-e \end{matrix}\right], \end{aligned} \tag{5.2.3}$$

where for convergence, by (5.2.2),

$$1 + \operatorname{Re}(2a - b - c - d - e) > 0. \tag{5.2.4}$$

We need one further result, namely, the bilateral binomial theorem. If a and c are complex numbers with $\operatorname{Re}(c-a) > 1$ and if z is a complex number with $z = e^{i\theta}$, $0 < \theta < 2\pi$, then

$${}_1H_1 \left[\begin{matrix} a; \\ c; \end{matrix} z\right] = \frac{(1-z)^{c-a-1}}{(-z)^{c-1}} \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}. \tag{5.2.5}$$

It would seem that Ramanujan had discovered (5.2.5), but we are unaware of any mention of it by him in his papers or notebooks. We remark that the bilateral binomial theorem can also be recovered from another bilateral hypergeometric series identity [11, p. 110, Theorem 2.8.2] due to Dougall [113], namely,

$$ {}_2H_2\left[\begin{matrix} a,\ b; \\ c,\ d; \end{matrix}\ 1\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{5.2.6} $$

where $\mathrm{Re}(c+d-a-b) > 1$ for convergence. In fact, in the identity above, first replacing b by dz and second, letting $d \to +\infty$, we derive (5.2.5) in view of Stirling's asymptotic formula for the Γ-function.

The first appearance of (5.2.5) of which we are aware is in T.H. Koornwinder's paper [187, p. 91 (middle of the page)] in 1994. When the second author and W. Chu gave their proof of Entry 5.1.2 in [50], they used a formulation of (5.2.5) given by M.E. Horn [164] in 2003. His original formulation is incorrect, but it is corrected in the proof by J.M. Borwein, which follows the statement of the problem, and indeed the correct version (5.2.5) was used by Berndt and Chu in [50]. In addition to the proof accompanying the original problem, another proof published on the aforementioned website [164] is by G.C. Greubel.

In the sequel, we very often use the classical reflection formula

$$ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{5.2.7} $$

5.3 Proof of Entry 5.1.1

We show that (5.2.3) leads to a proof of Entry 5.1.1.

Proof. Let S denote the series on the left-hand side of (5.1.1). Define

$$ \Omega := \frac{\sin\{\pi(\beta-\xi)\}\sin\{\pi(\delta-\xi)\}\sin\{\pi\alpha\}\sin\{\pi\gamma\}}{\pi^4}. \tag{5.3.1} $$

Using (5.2.7) and (5.3.1), we see that we can write S in the form

$$ \begin{aligned} S &= \Omega\xi \sum_{n=-\infty}^{\infty} \frac{(\xi+2n)\Gamma(1+\xi+n-\beta)\Gamma(1+\xi+n-\delta)\Gamma(1+n-\alpha)}{\xi\Gamma(\alpha+\xi+n)\Gamma(\gamma+\xi+n)\Gamma(\beta+n)\Gamma(\delta+n)} \\ &\quad \times \Gamma(1+n-\gamma) \\ &= \Omega\xi \frac{\Gamma(1+\xi-\beta)\Gamma(1+\xi-\delta)\Gamma(1-\alpha)\Gamma(1-\gamma)}{\Gamma(\alpha+\xi)\Gamma(\gamma+\xi)\Gamma(\beta)\Gamma(\delta)} \\ &\quad \times \sum_{n=-\infty}^{\infty} \frac{(1+\frac{1}{2}\xi)_n(1-\alpha)_n(1+\xi-\beta)_n(1-\gamma)_n(1+\xi-\delta)_n}{(\frac{1}{2}\xi)_n(\alpha+\xi)_n(\beta)_n(\gamma+\xi)_n(\delta)_n}. \end{aligned} \tag{5.3.2} $$

Note that the series (5.3.2) is well-poised, and so we can invoke (5.2.3) with $a = \xi$, $b = 1 - \alpha$, $c = 1 + \xi - \beta$, $d = 1 - \gamma$, and $e = 1 + \xi - \delta$. Thus, for $\mathrm{Re}(\alpha + \beta + \gamma + \delta) > 3$ for convergence, we deduce that

$$
\begin{aligned}
S &= \Omega\xi \frac{\Gamma(1-\alpha)\Gamma(1+\xi-\beta)\Gamma(1-\gamma)\Gamma(1+\xi-\delta)}{\Gamma(\alpha+\xi)\Gamma(\beta)\Gamma(\gamma+\xi)\Gamma(\delta)} \\
&\quad \times \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)} \\
&\quad \times \frac{\Gamma(\alpha+\xi)\Gamma(\beta-\xi)\Gamma(\gamma+\xi)\Gamma(\delta-\xi)\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(1+\xi)\Gamma(1-\xi)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} \\
&= \frac{\sin(\pi\xi)\Gamma(\alpha+\beta+\gamma+\delta-3)}{\pi\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)} \\
&\quad \times \frac{1}{\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},
\end{aligned}
$$

where we applied (5.2.7) five times, used the value of Ω from (5.3.1), and simplified. □

5.4 Proof of Entry 5.1.2

We first replace the functions in Entry 5.1.2 by another pair with which it is easier to work. With four applications of (5.2.7), we see that we can write $\varphi_s(\theta)$ in the form

$$
\varphi_s(\theta) = \frac{e^{si\theta} H_s(\theta)}{\Gamma(\alpha+s)\Gamma(\beta-s)\Gamma(\gamma+s)\Gamma(\delta-s)}, \tag{5.4.1}
$$

where

$$
H_s(\theta) := {}_2H_2 \left[\begin{matrix} 1-\beta+s, & 1-\delta+s; \\ \alpha+s, & \gamma+s; \end{matrix} \, e^{i\theta} \right]. \tag{5.4.2}
$$

Thus, we prove an analogue with φ_s and φ_t replaced by H_s and H_t, respectively. At the end of our proof, we convert our result to (5.1.2).

For brevity, we introduce the notation

$$
\langle s \rangle_n := \frac{(1-\beta+s)_n(1-\delta+s)_n}{(\alpha+s)_n(\gamma+s)_n}.
$$

In particular, we can then write

$$
H_s(\theta) = {}_2H_2 \left[\begin{matrix} 1-\beta+s, & 1-\delta+s; \\ \alpha+s, & \gamma+s; \end{matrix} \, e^{i\theta} \right] = \sum_{n=-\infty}^{\infty} \langle s \rangle_n e^{in\theta}.
$$

Proof. By the quotient rule for derivatives,

$$\frac{d}{d\theta}\left\{\frac{H_s(\theta)}{H_t(\theta)}e^{(s-t)i\theta}\right\}=\frac{\Delta}{e^{2ti\theta}H_t^2(\theta)}, \tag{5.4.3}$$

where

$$\Delta=e^{ti\theta}H_t(\theta)\frac{d}{d\theta}\left\{e^{si\theta}H_s(\theta)\right\}-e^{si\theta}H_s(\theta)\frac{d}{d\theta}\left\{e^{ti\theta}H_t(\theta)\right\}. \tag{5.4.4}$$

Using the notation above and in the previous paragraph and setting $k=m+n$ in the second equality below, we find that

$$\begin{aligned}\Delta &= i\sum_{m,n=-\infty}^{\infty}(s-t+n-m)\,\langle s\rangle_n\,\langle t\rangle_m\,e^{(s+t+n+m)i\theta}\\ &= i\sum_{k,n=-\infty}^{\infty}(s-t-k+2n)\,\langle s\rangle_n\,\langle t\rangle_{k-n}\,e^{(s+t+k)i\theta}\\ &= i\sum_{k=-\infty}^{\infty}(s-t-k)\,\langle t\rangle_k\,e^{(s+t+k)i\theta}\sum_{n=-\infty}^{\infty}\frac{s-t-k+2n}{s-t-k}\langle s\rangle_n\langle k+t\rangle_{-n}.\end{aligned} \tag{5.4.5}$$

Observe that the inner sum above is a well-poised ${}_5H_5$, requiring that $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>3$ for convergence. Thus, we can use (5.2.3) to obtain the evaluation

$$\begin{aligned}&{}_5H_5\left[\begin{matrix}1+\frac{1}{2}(s-t-k), & 1-\alpha-t-k, & 1-\beta+s, & 1-\gamma-t-k, & 1-\delta+s; \\ \frac{1}{2}(s-t-k), & \alpha+s, & \beta-t-k, & \gamma+s, & \delta-t-k;\end{matrix}\;1\right]\\ &=\Gamma\left[\begin{matrix}\alpha+t+k, & \gamma+t+k, & \beta-t-k, & \delta-t-k \\ 1+s-t-k, & 1-s+t+k, & \alpha+\gamma+s+t+k-1, & \beta+\delta-s-t-k-1\end{matrix}\right]\\ &\quad\times\;\Gamma\left[\begin{matrix}\alpha+s, & \beta-s, & \gamma+s, & \delta-s, & \alpha+\beta+\gamma+\delta-3 \\ \alpha+\beta-1, & \beta+\gamma-1, & \gamma+\delta-1, & \delta+\alpha-1\end{matrix}\right].\end{aligned} \tag{5.4.6}$$

Using the evaluation (5.4.6) in (5.4.5) and simplifying the expressions involving gamma functions and rising factorials, we find that

$$\begin{aligned}\Delta &= i\Gamma\left[\begin{matrix}\alpha+t, & \beta-t, & \gamma+t, & \delta-t \\ s-t, & 1-s+t, & \alpha+\gamma+s+t-1, & \beta+\delta-s-t-1\end{matrix}\right]\\ &\quad\times\Gamma\left[\begin{matrix}\alpha+s, \beta-s, \gamma+s, \delta-s, \alpha+\beta+\gamma+\delta-3 \\ \alpha+\beta-1, \beta+\gamma-1, \gamma+\delta-1, \delta+\alpha-1\end{matrix}\right]\\ &\quad\times e^{i(s+t)\theta}\sum_{k=-\infty}^{\infty}\frac{(s+t-\beta-\delta+2)_k}{(s+t+\alpha+\gamma-1)_k}e^{ik\theta}.\end{aligned} \tag{5.4.7}$$

We next apply Koornwinder's bilateral binomial theorem (5.2.5) with $a=s+t-\beta-\delta+2$ and $b=s+t+\alpha+\gamma-1$, subject to the condition $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>4$. Thus,

$$\sum_{k=-\infty}^{\infty} \frac{(s+t-\beta-\delta+2)_k}{(s+t+\alpha+\gamma-1)_k} e^{ik\theta} = {}_1H_1 \left[\begin{matrix} s+t-\beta-\delta+2; \\ s+t+\alpha+\gamma-1; \end{matrix} \, e^{i\theta} \right]$$
$$= \left(-e^{i\theta}\right)^{2-\alpha-\gamma-s-t} \left(1-e^{i\theta}\right)^{\alpha+\beta+\gamma+\delta-4}$$
$$\times \frac{\Gamma(\alpha+\gamma+s+t-1)\Gamma(\beta+\delta-s-t-1)}{\Gamma(\alpha+\beta+\gamma+\delta-3)}. \tag{5.4.8}$$

Now substitute (5.4.8) into (5.4.7), use (5.2.7), and cancel common gamma function factors to arrive at

$$\Delta = e^{(s+t)i\theta} \left(-e^{i\theta}\right)^{2-\alpha-\gamma-s-t} \left(1-e^{i\theta}\right)^{\alpha+\beta+\gamma+\delta-4}$$
$$\times \, i\Gamma \left[\begin{matrix} \alpha+s,\ \beta-s,\ \gamma+s,\ \delta-s,\ \alpha+t,\ \beta-t,\ \gamma+t,\ \delta-t \\ s-t,\ 1-s+t,\ \alpha+\beta-1,\ \beta+\gamma-1,\ \gamma+\delta-1,\ \delta+\alpha-1 \end{matrix} \right]$$
$$= \frac{i}{\pi} \sin\{\pi(s-t)\} \left(2\sin\tfrac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}$$
$$\times \, \Gamma \left[\begin{matrix} \alpha+s,\ \beta-s,\ \gamma+s,\ \delta-s,\ \alpha+t,\ \beta-t,\ \gamma+t,\ \delta-t \\ \alpha+\beta-1,\ \beta+\gamma-1,\ \gamma+\delta-1,\ \delta+\alpha-1 \end{matrix} \right]. \tag{5.4.9}$$

Lastly, substituting (5.4.9) into (5.4.3) and then reformulating the result according to the relation (5.4.1) between $\varphi_t(\theta)$ and $H_t(\theta)$, we derive the identity

$$\frac{d}{d\theta}\left\{\frac{\varphi_s(\theta)}{\varphi_t(\theta)}\right\} = \frac{i\ \sin\{\pi(s-t)\} \left(2\sin\frac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}}{\pi\varphi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

which is (5.1.3). The proof is thus complete. □

Let $\phi_t(\theta) = e^{-ti\theta}\varphi_t(\theta)$. We end this section with Ramanujan's rendition of Entry 5.1.2 given by

$$\frac{d}{d\theta}\left\{\frac{\phi_s(\theta)}{\phi_t(\theta)}\right\} \tag{5.4.10}$$
$$= \frac{i\sin\{\pi(s-t)\} \left|2\sin\frac{\theta}{2}\right|^{\alpha+\beta+\gamma+\delta-4} e^{i(\alpha-\beta+\gamma-\delta+2s-2t)\{(\pi-\theta)/2+\pi[\theta/(2\pi)]\}}}{\pi\phi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}.$$

Note that Ramanujan's function $\phi_s(\theta)$ does not have the factor $e^{si\theta}$ in $\varphi_s(\theta)$, defined in (5.1.2). The second major difference between the two formulas is in the exponent of e on the right-hand sides. One would guess that $[x]$ in Ramanujan's exponent denotes the greatest integer less than or equal to x. The powers of $2\sin(\frac{1}{2}\theta)$ in both (5.1.3) and (5.4.10) are the same, except that Ramanujan has absolute values around $2\sin(\frac{1}{2}\theta)$. In conclusion, except for multiplicative expressions of absolute value equal to 1, the other parts of the formulas (5.4.10) and (5.1.3) are identical.

5.5 Background on Continued Fractions and Orthogonal Polynomials

Any set of polynomials $\{p_n(x)\}$ that is orthogonal with respect to a positive measure satisfies a three-term recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{5.5.1}$$

where α_n, β_n, and γ_n are real and $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$. Conversely, Farvard's theorem informs us that if a set of polynomials $\{p_n(x)\}$ satisfies (5.5.1) with α_n, β_n, and γ_n real and with $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$, then there is a positive measure $d\psi(x)$ such that [10, 17]

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\, d\psi(x) = \begin{cases} 0, & m \neq n, \\ \dfrac{\gamma_1 \cdots \gamma_n}{\alpha_0 \cdots \alpha_{n-1}} \displaystyle\int_{-\infty}^{\infty} d\psi(x), & m = n. \end{cases} \tag{5.5.2}$$

We next review some basic properties of continued fractions. For the continued fraction

$$b_0 + \frac{a_1}{b_1} \, + \, \frac{a_2}{b_2} \, + \cdots + \, \frac{a_n}{b_n} \, + \cdots,$$

the nth approximant f_n is given by

$$f_n = b_0 + \frac{a_1}{b_1} \, + \, \frac{a_2}{b_2} \, + \cdots + \, \frac{a_n}{b_n} =: \frac{U_n}{V_n}.$$

We call U_n and V_n the nth numerator and denominator, respectively, of the continued fraction. If we define $U_{-1} = 1$, $V_{-1} = 0$, $U_0 = b_0$, and $V_0 = 1$, then, for $n = 1, 2, 3, \ldots$, the recurrence relations

$$b_n U_{n-1} + a_n U_{n-2} = U_n, \qquad b_n V_{n-1} + a_n V_{n-2} = V_n, \tag{5.5.3}$$

are valid [312, p. 15], [218, p. 8]. Using the recurrence relations in (5.5.3) and mathematical induction, one can easily deduce the following equivalence transformation [312, p. 19].

Proposition 5.5.1. *Let $c_0 = 1$ and $c_i \neq 0$ for $i > 0$. Then the two continued fractions*

$$b_0 + \frac{a_1}{b_1} \, + \, \frac{a_2}{b_2} \, + \, \frac{a_3}{b_3} \, + \cdots$$

and

$$c_0 b_0 + \frac{c_0 c_1 a_1}{c_1 b_1} \, + \, \frac{c_1 c_2 a_2}{c_2 b_2} \, + \, \frac{c_2 c_3 a_3}{c_3 b_3} \, + \cdots$$

have the same sequence of approximants.

The continuous symmetric Hahn polynomials $P_n(x)$, defined in (5.1.9), satisfy a three-term recurrence relation [16]

$$xP_n(x) = \alpha_n P_{n+1}(x) + \gamma_n P_{n-1}(x), \tag{5.5.4}$$

where

$$\alpha_n = \frac{(n+2\beta)(n+2\alpha+2\beta-1)}{2(2n+2\alpha+2\beta-1)} \quad \text{and} \quad \gamma_n = \frac{n(n+2\alpha-1)}{2(2n+2\alpha+2\beta-1)}. \tag{5.5.5}$$

Hence, by (5.5.3) and Proposition 5.5.1, the continued fraction corresponding to the orthogonal polynomials $P_n(x)$ with $\gamma_0 = -1$ is given by

$$\begin{aligned}\chi(x) :&= \frac{1}{x} - \frac{\alpha_0\gamma_1}{x} - \frac{\alpha_1\gamma_2}{x} - \frac{\alpha_2\gamma_3}{x} - \cdots \\ &= \frac{1}{x} - \frac{1\cdot(2\alpha)(2\beta)}{4x(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{x(2\alpha+2\beta+3)} \\ &\quad - \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4x(2\alpha+2\beta+5)} - \cdots.\end{aligned} \tag{5.5.6}$$

In other words, $P_n(x)$ is the nth denominator of $\chi(x)$.

On the other hand, (5.5.2) along with (5.5.4) and (5.5.5) provides the L^2-norm of the continuous symmetric Hahn polynomials [16, p. 653], namely,

$$\int_{-\infty}^{\infty} [P_n(x;\alpha,\beta)]^2 W(x)\,dx = \frac{(1)_n(2\alpha)_n(\alpha+\beta-\frac{1}{2})_n}{(2\beta)_n(2\alpha+2\beta-1)_n(\alpha+\beta+\frac{1}{2})_n} W_I,$$

where

$$W_I = \int_{-\infty}^{\infty} W(x)\ dx = \sqrt{\pi}\frac{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}, \tag{5.5.7}$$

where $W(x)$ is defined by (5.1.10). The integral in (5.5.7) is a special case of Barnes's beta integral [22]. This particular evaluation was also established by Ramanujan [255], [267, pp. 53–58], and R. Roy [273] using Fourier transforms and Mellin transforms, respectively.

It follows from (5.5.7) that the normalized weight function of the continuous symmetric Hahn polynomials $P_n(x)$ is given by

$$W_{\mathbb{N}}(x) := \frac{\Gamma(\alpha+\beta+\frac{1}{2})|\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}. \tag{5.5.8}$$

Since [255], [267, p. 54]

$$\phi(\alpha,x) = \frac{|\Gamma(\alpha+ix)|^2}{\Gamma^2(\alpha)}, \tag{5.5.9}$$

Entry 5.1.5 is equivalent to the following entry.

Entry 5.5.1 (p. 327). *For $\alpha > 0$ and $\beta > 0$,*

$$\int_0^\infty \frac{W_{\mathbb{N}}(x)\,dx}{1+4s^2x^2} = \frac{1}{2}\,+\,\frac{2(2\alpha)(2\beta)s^2}{2\alpha+2\beta+1}\,+\,\frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{2\alpha+2\beta+3}\,+\cdots.$$

Entry 5.5.1 gives a representation for the Stieltjes transform of the weight function of the continuous symmetric Hahn polynomials in terms of a continued fraction. A more general continued fraction with five free parameters was found by M.E.H. Ismail, J. Letessier, G. Valent, and J. Wimp [165]. Using contiguous relations for generalized hypergeometric functions of the type ${}_7F_6$, they derived explicit representations for the associated Wilson polynomials and computed the corresponding continued fraction.

5.6 Background on the Hamburger Moment Problem

Let $\{\mu_n\}$, $0 \le n < \infty$, be a sequence of real numbers. The Hamburger moment problem seeks to find a bounded, nondecreasing function $\psi(x)$ on the interval $(-\infty, \infty)$ satisfying the equations

$$\mu_n = \int_{-\infty}^{\infty} x^n \; d\psi(x), \qquad n = 0, 1, 2, \ldots. \tag{5.6.1}$$

Throughout this section, it is assumed that a solution $\psi(x)$ of the Hamburger moment problem (5.6.1) is increasing on an infinite number of points. If this solution is unique, the moment problem is said to be determinate; otherwise, it is indeterminate.

For any solution $\psi(x)$ of the moment problem (5.6.1), let

$$I(z,\psi) := \int_{-\infty}^{\infty} \frac{d\psi(x)}{z-x}, \tag{5.6.2}$$

where $z \in \mathbb{H} := \{z : \operatorname{Im} z > 0\}$. The following two lemmas show that there is a one-to-one correspondence between the elements of a certain class of functions to which $I(z,\psi)$ belongs and those in the class of solutions $\psi(x)$ of the moment problem (5.6.1).

Lemma 5.6.1. [286, Theorem 2.1] *The function $I(z,\psi)$ is analytic,* $\operatorname{Im} I(z,\psi) \le 0$ *on $\mathbb{H}$, and*

$$I(z,\psi) \sim \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}, \qquad 0 < \epsilon \le \arg z \le \pi - \epsilon, \quad 0 < \epsilon < \pi/2, \tag{5.6.3}$$

where μ_n, $n \ge 0$, is defined by (5.6.1).

Lemma 5.6.2. [286, Theorem 2.1] *If* $F(z)$ *is analytic,* $\operatorname{Im} F(z) \le 0$ *on* $\mathbb{H}$*, and*

$$F(z) \sim \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}, \qquad 0 < \epsilon \le \arg z \le \pi - \epsilon, \quad 0 < \epsilon < \pi/2, \tag{5.6.4}$$

where μ_n*,* $n \ge 0$*, is defined by* (5.6.1)*, then there exists a unique solution* $\psi(x)$ *of the moment problem* (5.6.1) *such that* $F(z) = I(z, \psi)$*.*

The integral $I(z, \psi)$ is also closely related to a continued fraction.

Lemma 5.6.3. [286, Theorem 2.4] *There exists a function* $F(z)$ *such that* $F(z)$ *is analytic,* $\operatorname{Im} F(z) \le 0$*, and* $F(z)$ *has a representation* (5.6.4) *if and only if there exists an associated continued fraction*

$$b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_3 + z} - \cdots \tag{5.6.5}$$

such that $a_n > 0$*,* $n \ge 0$*,* $b_n \in \mathbb{R}$ *for* $n \ge 0$*, and*

$$F(z) = b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \cdots - \frac{a_n}{F_{n+1}(z) + z},$$

where $F_{n+1}(z)$ *is an arbitrary analytic function,* $\operatorname{Im} F_{n+1}(z) \le 0$*, and* $F_{n+1}(z) = o(z)$ *as* $z \to \infty$ *on* $\mathbb{H}$*.*

In fact, the nth approximant, say $q_n(z)/p_n(z)$, of the continued fraction (5.6.5) can be expanded in the form [286, p. 35]

$$\frac{q_n(z)}{p_n(z)} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \cdots + \frac{\mu_{2n-1}}{z^{2n}} + \frac{\mu'_{2n}}{z^{2n+1}} + \frac{\mu'_{2n+1}}{z^{2n+2}} + \cdots, \tag{5.6.6}$$

where μ_j, $0 \le j \le 2n-1$, is defined in (5.6.1). (The definitions of μ'_n can be found in [286, p. 35]. Because their definitions are somewhat complicated and are not important in the present context, we do not give them here.) As we have seen in Sect. 5.5, the denominators $p_n(z)$ comprise a set of orthogonal polynomials of degree n by (5.5.3), and by (5.5.1) and (5.5.2) in Farvard's theorem. Moreover, the orthogonality relation

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) \, d\psi(x) = \begin{cases} 0, & m \ne n, \\ h_n, & m = n, \end{cases} \tag{5.6.7}$$

is satisfied by the solution $\psi(x)$ of the moment problem (5.6.1), [286, p. 35].

Next, we state two lemmas that provide, respectively, a sufficient and a necessary condition for a unique solution to the moment problem (5.6.1).

Lemma 5.6.4. [286, Theorem 2.9] *The moment problem* (5.6.1) *is determinate if*

$$\sum_{n=0}^{\infty} |p_n(z)|^2$$

diverges at a nonreal number z*.*

Lemma 5.6.5. [286, Theorem 2.10] *If the moment problem* (5.6.1) *is determinate, then the associated continued fraction* (5.6.5) *converges for all complex numbers* z*.*

5.7 The First Proof of Entry 5.1.5

Using the lemmas in Sect. 5.6, we prove the following theorem.

Theorem 5.7.1. *Let* $\alpha > 0$*,* $\beta > 0$*, and let* z *be nonreal. Then*

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)\,dx}{z-x} = \frac{1}{z} - \frac{1(2\alpha)(2\beta)}{4z(2\alpha+2\beta+1)} - \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z(2\alpha+2\beta+3)} - \cdots,$$

where $W_{\mathbb{N}}(x)$ *is defined in* (5.5.8)*.*

Since

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)}{z-x}\,dx = 2z\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)}{z^2-x^2}\,dx,$$

Theorem 5.7.1 is equivalent to

$$\begin{aligned}\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)}{z^2-x^2}dx = \frac{1}{2z^2} &- \frac{2(2\alpha)(2\beta)}{4(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z^2(2\alpha+2\beta+3)} \\ &- \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4(2\alpha+2\beta+5)} - \cdots,\end{aligned} \tag{5.7.1}$$

from which Entry 5.1.5 or Entry 5.5.1 immediately follows after replacing z by $i/2s$. Therefore, Theorem 5.7.1 implies that Theorem 5.1.5 holds for every complex number s except when s is purely imaginary.

In order to complete the proof of Theorem 5.7.1, we need a lemma of Stieltjes that gives the power series representation of a continued fraction of the type in (5.6.5).

Lemma 5.7.1. [312, Theorem 53.1] *The coefficients in the* J*-fraction*

$$\frac{1}{b_1+z} - \frac{a_1}{b_2+z} - \frac{a_2}{b_3+z} - \cdots$$

and its power series expansion

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{z^{n+1}}$$

are connected by the relations

$$c_{p+q} = k_{0,p}k_{0,q} + a_1k_{1,p}k_{1,q} + a_1a_2k_{2,p}k_{2,q} + \cdots,$$

where $k_{0,0} = 1$, $k_{r,s} = 0$ *if* $r > s$, *and* $k_{r,s}$, *for* $s \geq r$, *is recursively given by the matrix equations*

$$\begin{pmatrix} k_{00} & 0 & 0 & 0 & \cdots \\ k_{01} & k_{11} & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \cdot \begin{pmatrix} b_1 & 1 & 0 & 0 & \cdots \\ a_1 & b_2 & 1 & 0 & \cdots \\ 0 & a_2 & b_3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} = \begin{pmatrix} k_{01} & k_{11} & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & \cdots \\ k_{03} & k_{13} & k_{23} & k_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Proof of Theorem 5.7.1. Note that the continued fraction in Theorem 5.7.1 is $\chi(z)$ in (5.5.6), the continued fraction corresponding to the continuous symmetric Hahn polynomials $P_n(z)$. In the case of $\chi(z)$, $a_n = \alpha_{n-1}\gamma_n > 0$ and $b_n = 0$ for $n \geq 1$. It is therefore easy to see that $k_{ij} = 0$ when $i+j$ is odd, and thus $c_{2n+1} = 0$ and $c_{2n} > 0$. Let $Q_n(z)$ be the nth numerator of $\chi(z)$. Then for positive real numbers c_{2n} obtained from Lemma 5.7.1,

$$\frac{Q_n(z)}{P_n(z)} = \frac{1}{z} + \frac{c_2}{z^3} + \cdots + \frac{c_{2n-2}}{z^{2n-1}} + \frac{c_{2n}}{z^{2n+1}} \cdots .$$

Consider the moment problem

$$c_n = \int_{-\infty}^{\infty} x^n \, d\psi(x) \quad (n = 0, 1, 2, \dots) \tag{5.7.2}$$

for the sequence of real numbers c_n given above.

Observe that $P_0(\beta i) = 1$ and that more generally, by the Chu–Vandermonde theorem [11, p. 67, Corollary 2.2.3], $P_{4n}(\beta i) = 1$ for $n \geq 0$. Hence,

$$\sum_{n=0}^{\infty} |P_{4n}(\beta i)|^2$$

diverges, and thus the moment problem (5.7.2) has a unique solution $W_{\mathbb{N}}(x)$ by (5.6.7) and Lemma 5.6.4. It now follows from Lemmas 5.6.1–5.6.3, Lemma 5.6.5, and (5.6.6) that the continued fraction $\chi(z)$ converges to $I(z, \psi)$, for every nonreal number z, where $d\psi(x) = W_{\mathbb{N}}(x)dx$. □

5.8 The Second Proof of Entry 5.1.5

Recalling the definition of $\phi(\alpha, x)$ from either (5.1.7) or (5.5.9), set, for $t > 0$,

$$\Phi(\alpha, \beta, t) := \int_0^{\infty} \phi(\alpha, x)\phi(\beta, x)\cos(tx)\, dx. \tag{5.8.1}$$

Then, with the use of the elementary evaluation, for $x > 0$ and $s > 0$,

$$\int_0^{\infty} \cos(xt)e^{-st} \, dt = \frac{s}{s^2 + x^2},$$

the integral in Entry 5.1.5 can be rewritten in the form

$$\begin{aligned}\mathbb{I} :&= \int_0^\infty \phi(\alpha,x)\phi(\beta,x)\left(\frac{1}{2s}\int_0^\infty e^{-t/(2s)}\cos(xt)\,dt\right)\,dx \\ &= \frac{1}{2s}\int_0^\infty e^{-t/(2s)}\left(\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\cos(tx)\,dx\right)\,dt \\ &= \frac{1}{2s}\int_0^\infty e^{-t/(2s)}\Phi(\alpha,\beta,t)\,dt, \qquad (5.8.2)\end{aligned}$$

where we inverted the order of integration by absolute convergence.

Ramanujan [255], [267, p. 53] showed that by integrating termwise the partial fraction decomposition of the integrand,

$$\int_0^\infty \phi(\alpha,x)\cos(yx)\,dx = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right), \qquad y>0.$$

Hence, from the theory of Fourier cosine transforms,

$$\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\cos(xy)\,dy = \sqrt{\pi}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac12)}\phi(\alpha,x), \qquad x>0. \qquad (5.8.3)$$

Consequently,

$$\phi(\alpha,x) = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\cos(xy)\,dy. \qquad (5.8.4)$$

Applying (5.8.4) to (5.8.1), we deduce that

$$\Phi(\alpha,\beta,t) = \frac{1}{\pi}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\,\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta)}\,\mathcal{T}, \qquad (5.8.5)$$

where $\mathcal{T}$ is the triple integral

$$\mathcal{T} := \int_0^\infty\int_0^\infty\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{z}{2}\right)\cos(xy)\cos(xz)\cos(tx)\,dz\,dy\,dx. \qquad (5.8.6)$$

Using the elementary trigonometric identity $2\cos(xy)\cos(xz) = \cos(y+z)x + \cos(y-z)x$, replacing $-z$ by z in the integral involving $\cos(y-z)x$, setting $u = y+z$ in the second equality, inverting the order of integration with respect to x and y, and then replacing $-u$ by u in the integral over $-\infty < u \le 0$, we find that

$$\begin{aligned}\mathcal{T} &= \frac12\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{z}{2}\right)\cos((y+z)x)\cos(tx)\,dz\,dy\,dx \\ &= \frac12\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right)\cos(ux)\cos(tx)\,du\,dy\,dx \\ &= \frac12\int_0^\infty\int_0^\infty\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+u}{2}\right)\right. \\ &\qquad \left. + \operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right)\right)\cos(ux)\cos(tx)\,du\,dx\,dy. \qquad (5.8.7)\end{aligned}$$

Utilize the Fourier integral formula [305, p. 2]

$$\int_0^\infty \cos(nx)\,dx \int_0^\infty f(u)\cos(ux)\,du = \frac{\pi}{2} f(n)$$

in (5.8.7) to deduce that

$$\mathcal{T} = \frac{\pi}{4}\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right)+\operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right)\,dy. \tag{5.8.8}$$

In summary, so far, we have shown from (5.8.1), (5.8.5), and (5.8.8) that

$$\begin{aligned}\Phi(\alpha,\beta,t) &= \frac{1}{4}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)} \\ &\quad\times\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right)+\operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right)\,dy.\end{aligned} \tag{5.8.9}$$

The equality (5.8.9), which is a generalization of the integral of $W(x)$ in (5.5.7), was established also in [38, p. 226] as a consequence of Parseval's theorem, (5.8.3) above, and Legendre's duplication formula. In [38, p. 226], it was mentioned that M.L. Glasser [124] evaluated integrals like that on the right side in (5.8.9). Glasser used contour integration, but we use the binomial theorem and Euler's beta integral below.

Using the elementary identity

$$\begin{aligned}\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right)+\operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right) &= \operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{y}{2}\right) \\ \times\left\{\left(\frac{1}{1+\tanh\left(\frac{1}{2}y\right)\tanh\left(\frac{1}{2}y\right)}\right)^{2\beta}\right. &+ \left.\left(\frac{1}{1-\tanh\left(\frac{1}{2}y\right)\tanh\left(\frac{1}{2}y\right)}\right)^{2\beta}\right\}\end{aligned}$$

and the binomial theorem in (5.8.8), we find that

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\int_0^\infty \operatorname{sech}^{2\alpha+2\beta}\left(\frac{y}{2}\right)\sum_{n=0}^\infty \frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\tanh^{2n}\left(\frac{y}{2}\right)dy. \tag{5.8.10}$$

Setting $v=\tanh^2(\frac{1}{2}y)$ in (5.8.10), we arrive at

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^\infty \frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\int_0^1 (1-v)^{\alpha+\beta-1}\, v^{n-1/2}\,dv. \tag{5.8.11}$$

Using Euler's beta integral $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, we deduce that

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\frac{\Gamma(n+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n+\frac{1}{2})} \tag{5.8.12}$$

$$= \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}(\frac{1}{2})_n}{(2n)!(\alpha+\beta+\frac{1}{2})_n}\tanh^{2n}\left(\frac{t}{2}\right)$$

$$= \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\,{}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right).$$

Set $w=\tanh\left(\frac{1}{4}t\right)$, so that

$$F(t) := \operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\,{}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right)$$

$$= \left(\frac{1-w^2}{1+w^2}\right)^{2\beta}\,{}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\frac{4w^2}{(1+w^2)^2}\right). \tag{5.8.13}$$

Using the quadratic transformation [11, p. 128, Eq. (3.1.9)]

$${}_2F_1(a,b;a-b+1;z) = (1+z)^{-a}\,{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(1+z)^2}\right)$$

with $z=w^2$, $a=2\beta$, and $b=\beta-\alpha+\frac{1}{2}$, we find that

$$F(t) = (1-w^2)^{2\beta}\,{}_2F_1\left(-\alpha+\beta+\frac{1}{2},2\beta;\alpha+\beta+\frac{1}{2};w^2\right),$$

and using Pfaff's transformation formula [11, p. 68, Theorem 2.2.5]

$$(1-z)^a\,{}_2F_1(a,b;c;z) = {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

with $a=2\beta$, $b=-\alpha+\beta+\frac{1}{2}$, $c=\alpha+\beta+\frac{1}{2}$, and $z=w^2$, we find that

$$F(t) = {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{w^2}{w^2-1}\right). \tag{5.8.14}$$

Therefore, by (5.8.12)–(5.8.14),

$$\mathcal{T} = \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\,{}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right). \tag{5.8.15}$$

From (5.8.5) and (5.8.15), we now see that

$$\Phi(\alpha,\beta,t) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})}$$

$$\times\,{}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right). \tag{5.8.16}$$

Then, it follows from (5.8.2) and (5.8.16) that

$$\mathbb{I} = \frac{\sqrt{\pi}}{4s} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})} \times \int_0^\infty {}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; -\sinh^2\frac{t}{4}\right) e^{-t/(2s)}\, dt. \tag{5.8.17}$$

Recall the continued fraction expansion of Stieltjes [297], [298, pp. 282–291],

$$\int_0^\infty {}_2F_1\left(a, b; \frac{a+b+1}{2}; -\sinh^2 t\right) e^{-tz}\, dt = \frac{1}{z}\,+\,\frac{1\cdot ab\cdot 4}{(a+b+1)z}\,+\,\frac{2\cdot(a+1)(b+1)(a+b)\cdot 4}{(a+b+3)z} + \frac{3\cdot(a+2)(b+2)(a+b+1)\cdot 4}{(a+b+5)z}\,+\cdots, \qquad \operatorname{Re} z>0, \tag{5.8.18}$$

from which, upon replacing t by $4t$ and setting $a=2\alpha$ and $b=2\beta$, we deduce that, for $s>0$,

$$\int_0^\infty {}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; -\sinh^2\frac{t}{4}\right) e^{-t/(2s)}\, dt \tag{5.8.19}$$
$$= \frac{4}{2/s}\,+\,\frac{1\cdot(2\alpha)(2\beta)\cdot 4}{(2\alpha+2\beta+1)2/s}\,+\,\frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)\cdot 4}{(2\alpha+2\beta+3)2/s}\,+\cdots.$$

By (5.8.17) and (5.8.19), we finally deduce that

$$\mathbb{I} = \sqrt{\pi}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})} \times\left(\frac{1}{2}\,+\,\frac{2\cdot 1\cdot(2\alpha)(2\beta)s^2}{(2\alpha+2\beta+1)}\,+\,\frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{(2\alpha+2\beta+3)}\,+\cdots\right),$$

which completes the proof of Entry 5.1.5 for $s>0$. By (5.5.7), Entry 5.1.5 holds for $s=0$. Since both sides of Theorem 5.1.5 are even functions of s, Theorem 5.1.5 is valid for all real s.

5.9 Proof of Entry 5.1.2

We use the recurrence relation (5.5.3) and induction to prove that for all integers $n\geq 1$,

$$f_n = \frac{1}{a+b+1}\,+\,\frac{1\cdot(a+1)(b+1)(a+b+1)}{a+b+3}\,+\cdots + \frac{n\cdot(a+n)(b+n)(a+b+n)}{a+b+(2n+1)} \tag{5.9.1}$$
$$= \frac{1}{a+b+1}(1-A_1+A_1A_2-A_1A_2A_3+\cdots+(-1)^n A_1A_2\cdots A_n) := R_n,$$

where A_n is defined in (5.1.6). Entry 5.1.2 then readily follows from (5.9.1).

First, observe from (5.1.6) that

$$A_t = \begin{cases} \dfrac{(a+t)(b+t)}{(a+1+t)(b+1+t)}, & \text{if } t \text{ is odd,} \\ \dfrac{t(a+b+t)}{(a+b+t+1)(t+1)}, & \text{if } t \text{ is even.} \end{cases} \tag{5.9.2}$$

From the recurrence relations (5.5.3), we find that

$$\begin{aligned} f_1 &= \frac{U_1}{V_1} = \frac{1}{a+b+1} = R_1, \\ f_2 &= \frac{U_2}{V_2} = \frac{a+b+3}{(a+b+1)(a+2)(b+2)} = \frac{1}{a+b+1}(1-A_1) = R_2, \\ f_3 &= \frac{U_3}{V_3} = \frac{3(a+b+3)^2+2(a+1)(b+1)(a+b+2)}{3(a+b+1)(a+b+3)(a+2)(b+2)} \\ &= \frac{1}{a+b+1}(1-A_1+A_1A_2) = R_3, \\ f_4 &= \frac{U_4}{V_4} = \frac{3(a+b+3)^2(a+4)(b+4)+2(a+1)(b+1)(a+b+2)(a+b+7)}{3(a+b+1)(a+b+3)(a+2)(b+2)(a+4)(b+4)} \\ &= \frac{1}{1+a+b}(1-A_1+A_1A_2-A_1A_2A_3) = R_4. \end{aligned}$$

Assume that (5.9.1) holds up to k. Then by (5.5.3),

$$f_{k+1} = \frac{U_{k+1}}{V_{k+1}} = \frac{(a+b+(2k+1))U_k + k(a+k)(b+k)(a+b+k)U_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}}.$$

By the induction hypothesis, the numerator above equals

$$\begin{aligned} &\frac{a+b+(2k+1)}{a+b+1}(1-A_1+A_1A_2+\cdots+(-1)^{k-1}A_1A_2\cdots A_{k-1})V_k \\ &+ \frac{k(a+k)(b+k)(a+b+k)}{a+b+1} \\ &\times (1-A_1+A_1A_2+\cdots+(-1)^{k-2}A_1A_2\cdots A_{k-2})V_{k-1}. \end{aligned}$$

Hence, we may write

$$\begin{aligned} f_{k+1} = &\frac{1}{a+b+1}(1-A_1+A_1A_2+\cdots+(-1)^{k-1}A_1A_2\cdots A_{k-1}) \\ &- \frac{(-1)^{k-1}A_1A_2\cdots A_{k-1}}{a+b+1} \\ &\times \frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}}. \end{aligned}$$

It therefore suffices to prove that

$$\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}} = A_k. \tag{5.9.3}$$

We claim that

$$V_k = \begin{cases} V_{k-1}(a+b+2k-1+(k-1)(a+b+k-1)), & \text{if } k \text{ is odd,} \\ V_{k-1}(a+b+2k-1+(a+k-1)(b+k-1)), & \text{if } k \text{ is even.} \end{cases} \tag{5.9.4}$$

We shall defer the proof of the claim above until the end of the proof (5.9.1).

Assuming the truth of (5.9.4) for the moment, let k be odd. Then the left side of (5.9.3) is equal to

$$\begin{aligned}
&\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{\begin{pmatrix}(a+b+2k+1)V_{k-1}\{a+b+2k-1+(k-1)(a+b+k-1)\} \\ +k(a+k)(b+k)(a+b+k)V_{k-1}\end{pmatrix}} \\
&= \frac{k(a+k)(b+k)(a+b+k)}{(a+b+2k+1)(ak+bk+k^2)+k(a+k)(b+k)(a+b+k)} \\
&= \frac{(a+k)(b+k)}{a+b+2k+1+(a+k)(b+k)} = \frac{(a+k)(b+k)}{(a+1+k)(b+1+k)} = A_k,
\end{aligned}$$

as desired. When k is even, the left-hand side of (5.9.3) takes the shape

$$\begin{aligned}
&\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{\begin{pmatrix}(a+b+2k+1)V_{k-1}\{a+b+2k-1+(a+k-1)(b+k-1)\} \\ +k(a+k)(b+k)(a+b+k)V_{k-1}\end{pmatrix}} \\
&= \frac{k(a+k)(b+k)(a+b+k)}{(a+b+2k+1)(a+k)(b+k)+k(a+k)(b+k)(a+b+k)} \\
&= \frac{k(a+b+k)}{(a+b+k+1)(k+1)} = A_k,
\end{aligned}$$

which again is what we wanted to prove. It remains to prove the claim.

We can recast (5.9.4) in the equivalent form

$$\frac{V_k}{V_{k-1}} = \begin{cases} (a+k)(b+k), & \text{if } k \text{ is even,} \\ k(a+b+k), & \text{if } k \text{ is odd.} \end{cases} \tag{5.9.5}$$

We now prove (5.9.5). The first few instances of (5.9.5) are

$$\begin{aligned}
\frac{V_2}{V_1} &= \frac{(a+2)(b+2)(a+b+1)}{a+b+1} = (a+2)(b+2), \\
\frac{V_3}{V_2} &= \frac{3(a+2)(b+2)(a+b+1)(a+b+3)}{(a+2)(b+2)(a+b+1)} = 3(a+b+3), \\
\frac{V_4}{V_3} &= (a+4)(b+4).
\end{aligned}$$

Assume that (5.9.5) is true up to $2k$. Then, by (5.5.3) and the induction hypothesis,

$$\begin{aligned}\frac{V_{2k+1}}{V_{2k}} &= \frac{(a+b+4k+1)V_{2k}+2k(a+2k)(b+2k)(a+b+2k)V_{2k-1}}{V_{2k}} \\ &= (a+b+4k+1)+2k(a+2k)(b+2k)(a+b+2k)\cdot\frac{1}{(a+2k)(b+2k)} \\ &= (2k+1)(a+b+2k+1),\end{aligned}$$

which is in agreement with (5.9.5). Assuming that (5.9.5) is valid up to $2k+1$ and using (5.5.3) again, we find, upon the use of the induction hypothesis, that

$$\begin{aligned}&\frac{V_{2k+2}}{V_{2k+1}} \\ &= (a+b+4k+3)V_{2k+1}+(2k+1)(a+2k+1)(b+2k+1)(a+b+2k+1) \\ &\quad\times\frac{V_{2k}}{V_{2k+1}} \\ &= (a+b+4k+3)+(2k+1)(a+2k+1)(b+2k+1)(a+b+2k+1) \\ &\quad\times\frac{1}{(2k+1)(a+b+2k+1)} \\ &= (a+2k+2)(b+2k+2),\end{aligned}$$

which again is in harmony with (5.9.5). This then completes the proof of Ramanujan's assertion in (5.9.1) and Entry 5.1.2. As mentioned in the introduction, this proof is due to S.-Y. Kang.

6

Two Partial Manuscripts on Euler's Constant γ

6.1 Introduction

Like many mathematicians, Ramanujan was evidently fascinated with Euler's constant γ. He wrote only one paper on Euler's constant [264], [267, pp. 163–168], but published with his lost notebook [269, pp. 274–277] are two partial manuscripts devoted to γ.

First, on pages 274 and 275 in [269], there is the beginning of a manuscript that probably was to focus on integrals related to Euler's constant γ and $\psi(s) := \Gamma'(s)/\Gamma(s)$, and on integrals and series related to Frullani's integral theorem [37, p. 313, Eq. (2.15)], [142]. This fragment contains only two short sections, comprising one and a half pages. Afterward, Ramanujan wrote "3." to indicate the beginning of a third section, but the manuscript ends abruptly at this point.

The second partial manuscript is related to the first problem that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [241], [267, p. 322] and to the first six entries of Chap. 2 in his second notebook [267], [37, pp. 25–35]. Moreover, the second partial manuscript gives Ramanujan's solution to another problem [243], [267, p. 325] that he submitted to the *Journal of the Indian Mathematical Society.* No solution to this problem was ever published in the *Journal of the Indian Mathematical Society.* The formula for γ in this problem was also recorded in Ramanujan's second notebook as Entry 16 of Chap. 8 [268], [37, p. 196]. In [37], we gave a solution based on material in Chap. 2 of Ramanujan's second notebook [268], [37, pp. 25–35], where he considers a more general series and derives several elegant theorems and examples. The solution that Ramanujan gives in his lost notebook is not fundamentally different from that given by the second author in [37], but since it is more self-contained and independent of our considerations in [37, pp. 25–35], for those readers not desiring to read the aforementioned material in Chap. 2 and only interested in a direct route to Ramanujan's formula for Euler's constant, we provide Ramanujan's solution in this chapter. We mildly

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correct Ramanujan's claim and give his proof while providing a few additional details. Lastly, we employ Ramanujan's formula to numerically calculate γ.

The proofs in this chapter were first published in papers that Berndt wrote with D. Bowman [46] and T. Huber [55].

6.2 Theorems on γ and $\psi(s)$ in the First Manuscript

We first prove the primary theorem in the first section of the first-mentioned incomplete manuscript. Applications of this result have been made by H. Alzer and S. Koumandos [7] in deriving series representations for γ, Catalan's constant, $\zeta(3)$, π^2, and other familiar constants.

Entry 6.2.1 (p. 274). *Let p, q, and r be positive. Then*

$$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx = \psi(q/r) - \psi(p) + \log r. \tag{6.2.1}$$

Proof. (Ramanujan) Using the continuity of the integrand on the right side below for $0 \leq x, s \leq 1$, a well-known integral representation for the beta function, the change of variable $t = x^r$ in the second part of the integrand, and L'Hospital's rule, we find that

$$\begin{aligned}
&\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx \\
&= \lim_{s \to 0+} \int_0^1 \left\{ x^{p-1}(1-x)^{s-1} - r^{1-s} x^{q-1}(1-x^r)^{s-1} \right\} dx \\
&= \lim_{s \to 0} \left\{ \frac{\Gamma(p)\Gamma(s)}{\Gamma(s+p)} - r^{-s} \frac{\Gamma(q/r)\Gamma(s)}{\Gamma(s+q/r)} \right\} \\
&= \lim_{s \to 0} \frac{\left\{ \dfrac{\Gamma(p)}{\Gamma(s+p)} - r^{-s} \dfrac{\Gamma(q/r)}{\Gamma(s+q/r)} \right\} \Gamma(s+1)}{s} \\
&= \lim_{s \to 0} \left\{ -\frac{\Gamma(p)\Gamma'(s+p)}{\Gamma^2(s+p)} + \Gamma(q/r) \left(\frac{r^{-s} \log r}{\Gamma(s+q/r)} + \frac{r^{-s}\Gamma'(s+q/r)}{\Gamma^2(s+q/r)} \right) \right\} \\
&= -\psi(p) + \log r + \psi(q/r),
\end{aligned}$$

which completes the proof. □

Entry 6.2.2 (p. 274). *Suppose that a, b, and c are positive with $b > 1$. Then*

$$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b} \right) \sum_{k=0}^{\infty} x^{ab^k} \, dx = \psi\left(\frac{a}{b} + c \right) - \log \frac{a}{b}.$$

Proof. By Entry 6.2.1 and the facts that $b > 1$ and $\psi(x) \sim \log x$, as x tends to ∞ (see (13.2.28)),

$$\begin{aligned}\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b} \right) \sum_{k=0}^n x^{ab^k} dx &= \sum_{k=0}^n \int_0^1 \left(\frac{x^{c+ab^k-1}}{1-x} - \frac{bx^{bc+ab^k-1}}{1-x^b} \right) dx \\ &= \sum_{k=0}^n \left(\psi\left(ab^{k-1}+c\right) - \psi\left(ab^k+c\right) + \log b \right) \\ &= \psi\left(\frac{a}{b}+c\right) - \psi\left(ab^n+c\right) + (n+1)\log b \\ &= \psi\left(\frac{a}{b}+c\right) - \log\left(ab^n+c\right) + (n+1)\log b + o(1) \\ &= \psi\left(\frac{a}{b}+c\right) - n\log b - \log a + (n+1)\log b + o(1) \\ &= \psi\left(\frac{a}{b}+c\right) - \log\frac{a}{b} + o(1),\end{aligned}$$

as n tends to ∞. Letting $n \to \infty$, we complete the proof. □

Entry 6.2.3 (p. 275). *We have*

$$\int_0^1 \frac{1}{1+x} \sum_{k=1}^\infty x^{2^k} dx = 1 - \gamma, \tag{6.2.2}$$

$$\int_0^1 \frac{1+2x}{1+x+x^2} \sum_{k=1}^\infty x^{3^k} dx = 1 - \gamma, \tag{6.2.3}$$

$$\int_0^1 \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})(1+\sqrt{x}+x)} \sum_{k=1}^\infty x^{(3/2)^k} dx = 1 - \gamma. \tag{6.2.4}$$

Proof. In Entry 6.2.2, set, respectively, $c = 1$, $a = b = 2$; $c = 1$, $a = b = 3$; and $c = 1$, $a = b = 3/2$. Use the fact that [126, p. 954]

$$\psi(2) = 1 - \gamma \tag{6.2.5}$$

to complete the proof. □

According to Bromwich [80, p. 526], (6.2.2) is due to E. Catalan. Parts (6.2.3) and (6.2.4) may be new. H. Alzer and S. Koumandos [8] have employed (6.2.2) in deriving further representations for γ; several references to the literature on γ can be found in [8].

Before discussing the very brief second section of Ramanujan's fragment, we offer some alternative proofs, references, and connections with further work of Ramanujan, as well as others.

Lemma 6.2.1. *For $x > 0$, $x \neq 1$, and any integer $n > 1$,*

$$\frac{1}{\log x} + \frac{1}{1-x} = \sum_{k=1}^\infty \frac{(n-1) + (n-2)x^{1/n^k} + (n-3)x^{2/n^k} + \cdots + x^{(n-2)/n^k}}{n^k\left(1 + x^{1/n^k} + x^{2/n^k} + \cdots + x^{(n-1)/n^k}\right)}. \tag{6.2.6}$$

Proof. It is easy to verify that

$$\frac{1}{1-x^n} = \frac{1}{n}\left(\frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}} + \frac{1}{1-x}\right). \tag{6.2.7}$$

Replacing x by $x^{1/n}$ and iterating m times, we find that

$$\frac{1}{1-x} = \sum_{k=1}^{m} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})} + \frac{1}{n^m(1-x^{1/n^m})}.$$

If we now let m tend to ∞ and apply L'Hospital's rule, we complete the proof. □

The special cases $n = 2, 3$ of Lemma 6.2.1 can be found in Ramanujan's third notebook [268, p. 364], and proofs can be found in Berndt's book [40, pp. 399–400]. Our proof here generalizes these proofs.

Lemma 6.2.2. *For every integer* $n > 1$,

$$\gamma = \int_0^1 \left(\frac{n}{1-x^n} - \frac{1}{1-x}\right) \sum_{k=1}^{\infty} x^{n^k-1} dx. \tag{6.2.8}$$

Proof. Integrate (6.2.6) over $0 \le x \le 1$ and employ the well-known integral representation [80, p. 507], [126, p. 955]

$$\gamma = \int_0^1 \left(\frac{1}{\log x} + \frac{1}{1-x}\right) dx.$$

Accordingly, replacing x by x^{n^k}, we find that

$$\begin{aligned}
\gamma &= \int_0^1 \sum_{k=1}^{\infty} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})} dx \\
&= \sum_{k=1}^{\infty} \int_0^1 \frac{1}{n^k} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k}} dx \\
&= \sum_{k=1}^{\infty} \int_0^1 \frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}} x^{n^k-1} dx \\
&= \int_0^1 \left(\frac{n}{1-x^n} - \frac{1}{1-x}\right) \sum_{k=1}^{\infty} x^{n^k-1} dx,
\end{aligned}$$

by (6.2.7). This completes the proof. □

Lemma 6.2.2 is equivalent to Entry 6.2.2 in the case $c = 1, a = b = n$. To see this, first make these substitutions in Entry 6.2.2 and use (6.2.5) to deduce that

$$1 - \gamma = \int_0^1 \left(\frac{1}{1-x} - \frac{nx^{n-1}}{1-x^n} \right) \sum_{k=1}^{\infty} x^{n^k} dx. \tag{6.2.9}$$

Adding (6.2.8) and (6.2.9) and simplifying, we readily find that

$$1 = (n-1) \int_0^1 \sum_{k=1}^{\infty} x^{n^k - 1} dx,$$

which is trivially verified by termwise integration.

The arguments in the proof of Lemma 6.2.2 lead to another formula for γ. A proof of this formula can be found in the paper by Berndt and Bowman [46] and in the Master's Thesis of C.S. Haley [140].

Theorem 6.2.1. *If b is an integer exceeding 1, let*

$$\epsilon_r = \begin{cases} b-1, & \text{if } b \mid r, \\ -1, & \text{if } b \nmid r. \end{cases} \tag{6.2.10}$$

Then

$$\gamma = \sum_{r=1}^{\infty} \frac{\epsilon_r}{r} \left[\frac{\log r}{\log b} \right],$$

where $[x]$ denotes the greatest integer $\leq x$.

Corollary 6.2.1. *We have*

$$\gamma = \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left[\frac{\log r}{\log 2} \right]. \tag{6.2.11}$$

Proof. Let $b = 2$ in Theorem 6.2.1. □

The representation for γ given in (6.2.11) was discovered in 1909 by G. Vacca [307] and is known as Dr. Vacca's series for γ. Corollary 6.2.1 was rediscovered by H.F. Sandham, who submitted it as a problem [274]. M. Koecher [185] obtained a generalization of (6.2.11) that includes a formula for γ submitted by Ramanujan as a problem [243], [267, p. 325] to the *Journal of the Indian Mathematical Society*, and found in his notebooks [268], [37, p. 196]. Further series in the spirit of those of Ramanujan and Koecher were found by F.L. Bauer [25]. A result similar to that of Bauer was found by A.W. Addison [2], with a simpler version later established by I. Gerst [121]. For alternative versions of Vacca's series for γ, for generalizations, and for approximations to γ, see papers by J. Sondow [293], Sondow and W. Zudilin [294], and Kh. Hessami Pilehrood and T. Hessami Pilehrood [154–156].

J.W.L. Glaisher [123] generalized Theorem 6.2.1. We offer a theorem that is equivalent to his theorem. For a proof, we refer to the paper by Berndt and Bowman [46]. Another proof has been found by Haley [140].

Theorem 6.2.2. *Let a and b be positive integers with $b > 1$, and let ϵ_r be defined by* (6.2.10). *Then*

$$\log a + \gamma - \sum_{n=1}^{a-1} \frac{1}{n} = \sum_{r=a}^{\infty} \frac{\epsilon_r}{r} \left[\frac{\log(r/a)}{\log b} \right].$$

We complete this section with a remark about Entry 6.2.1. After replacing x by e^{-x} in (6.2.1), we obtain an integral of Frullani type. In his third quarterly report, Ramanujan found a beautiful generalization of Frullani's theorem. In particular, the formula

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} dx = \psi(q) - \psi(p) + \log \frac{b}{a}, \qquad (6.2.12)$$

where $a, b, p, q > 0$, is a special instance of Ramanujan's theorem [37, p. 314]. In view of the right sides of (6.2.1) and (6.2.12), one might surmise that (6.2.1) can be derived from (6.2.12), or Ramanujan's generalization of Frullani's theorem, and this was accomplished by J.-P. Allouche [3].

6.3 Integral Representations of $\log x$

Section 2 in Ramanujan's first unpublished fragment is devoted solely to the statements of the following theorem and (6.3.1) below.

Entry 6.3.1 (p. 275). *If a, b, and c are positive with $b > 1$, then*

$$\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{\infty} x^{ab^k} dx = -\log\left(1 + \frac{bc}{a}\right).$$

Proof. As indicated by Ramanujan, we begin with the equality [126, p. 575]

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\log x} dx = -\log \frac{q}{p}, \qquad (6.3.1)$$

where $p, q > 0$. Thus, since $b > 1$,

$$\begin{aligned}
-\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{n} x^{ab^k} dx &= \sum_{k=0}^{n} \int_0^1 \frac{x^{c+ab^k-1} - x^{bc+ab^k-1}}{\log x} dx \\
&= \sum_{k=0}^{n} \log \frac{bc + ab^k}{c + ab^k} \\
&= \sum_{k=0}^{n} \left(\log b + \log(c + ab^{k-1}) - \log(c + ab^k) \right) \\
&= (n+1) \log b + \log(c + a/b) - \log(c + ab^n) \\
&= (n+1) \log b + \log(c + a/b) - n \log b - \log a + o(1) \\
&= \log(1 + bc/a) + o(1),
\end{aligned}$$

as n tends to ∞. Letting n tend to ∞, we complete the proof. □

Entry 6.3.2 (p. 275). *We have*

$$\int_0^1 \frac{1-x}{\log x} \sum_{k=1}^{\infty} x^{2^k}\,dx = -\log 2.$$

Proof. Set $c = 1$ and $a = b = 2$ in Entry 6.3.1. □

Observe that if x is replaced by e^{-x} in (6.3.1), we obtain an example of Frullani's integral theorem. Ramanujan's ideas can be extended to other examples of Frullani-type integrals found by, among others, Ramanujan in his quarterly reports [37] and Hardy [142], [151, pp. 195–226]. For example, we note the integral [142, Eq. (29)], [267, p. 200]

$$\int_0^{\infty} \frac{e^{-ax}\cos(\alpha x) - e^{-bx}\cos(\beta x)}{x}\,dx = -\frac{1}{2}\log\frac{a^2+\alpha^2}{b^2+\beta^2}, \tag{6.3.2}$$

where $a, b, \alpha, \beta > 0$.

6.4 A Formula for γ in the Second Manuscript

At the top of page 276 in [269], Ramanujan writes

$$\begin{aligned}\gamma = \log 2 - \frac{2}{3^3-3} - 4\left(\frac{1}{6^3-6} + \frac{1}{9^3-9} + \frac{1}{12^3-12}\right)\\ - 6\left(\frac{1}{15^3-15} + \frac{1}{18^3-18} + \cdots + \frac{1}{39^3-39}\right) - \cdots,\\ \text{the last term of the } n\text{th group being } \frac{1}{\left(\frac{3^n+3}{2}\right)^3} - \frac{1}{\frac{3^n+3}{2}}.\end{aligned} \tag{6.4.1}$$

Ramanujan's assertion (6.4.1) needs to be slightly corrected. The *first*, not the last, term of the nth group is $\dfrac{1}{\left(\frac{3^n+3}{2}\right)^3} - \dfrac{1}{\frac{3^n+3}{2}}$. We give a more precise statement of Ramanujan's claim.

Entry 6.4.1 (p. 276).

$$\gamma = \log 2 - \sum_{n=1}^{\infty} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{6.4.2}$$

Proof. It is easily checked that for each positive integer k,

$$\frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1} = \frac{1}{k} + \frac{2}{(3k)^3 - 3k}. \tag{6.4.3}$$

Set $k = 1, 2, \ldots, n$ in (6.4.3) and add the n equalities to find that

$$\sum_{k=2}^{3n+1} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{2}{(3k)^3 - 3k},$$

i.e.,

$$\sum_{k=1}^{2m+1} \frac{1}{m+k} = 1 + \sum_{k=1}^{m} \frac{2}{(3k)^3 - 3k}. \tag{6.4.4}$$

The first three cases, $m = 1, 2, 3$, of (6.4.4) are, respectively,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + \frac{2}{3^3 - 3},$$

$$\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{13} = 1 + \frac{2}{3^3 - 3} + \frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12},$$

$$\frac{1}{4} + \frac{1}{15} + \cdots + \frac{1}{40} = 1 + \frac{2}{3^3 - 3} + \cdots + \frac{2}{39^3 - 39}.$$

More generally, taking $m = 1, 2, \ldots, n$ in (6.4.4) and adding the n equalities, we find that

$$\sum_{k=1}^{\frac{3^n-1}{2}} \frac{1}{k} = n + (n-1)\frac{2}{3^3 - 3} + (n-2)\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right)$$
$$+ (n-3)\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \cdots + \frac{2}{39^3 - 39}\right), \tag{6.4.5}$$

where there are n expressions on the right-hand side of (6.4.5). Now, from the standard definition of Euler's constant, as $n \to \infty$,

$$\sum_{k=1}^{\frac{3^n-1}{2}} \frac{1}{k} = \log\left(\frac{3^n - 1}{2}\right) + \gamma + o(1) = n \log 3 - \log 2 + \gamma + o(1). \tag{6.4.6}$$

If we use (6.4.6) in (6.4.5), divide both sides of the resulting equality by n, and then let $n \to \infty$, we deduce that

$$\log 3 = 1 + \sum_{k=1}^{\infty} \frac{2}{(3k)^3 - 3k}. \tag{6.4.7}$$

(The identity (6.4.7) is also found in Sect. 2 of Chap. 2 in Ramanujan's second notebook [268]; see also [37, p. 27].) Lastly, using (6.4.6) in (6.4.5), letting $n \to \infty$ while invoking (6.4.7), and rearranging, we readily arrive at (6.4.2) to complete the proof. □

6.5 Numerical Calculations

Define

$$S_j := \sum_{n=1}^{j} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{6.5.1}$$

The first 14 values of $-\gamma + \log 2 - S_j$ are given in the following table.

j	$-\gamma + \log 2 - S_j$	j	$-\gamma + \log 2 - S_j$
1	3.25982×10^{-2}	8	3.14043×10^{-8}
2	5.66401×10^{-3}	9	3.87176×10^{-9}
3	8.37419×10^{-4}	10	4.72684×10^{-10}
4	1.15710×10^{-4}	11	5.72414×10^{-11}
5	1.53668×10^{-5}	12	6.88472×10^{-12}
6	1.98621×10^{-6}	13	8.23230×10^{-13}
7	2.51665×10^{-7}	14	6.05812×10^{-14}

These calculations were carried out using Mathematica 5.2. The partial sums in (6.5.1) are taken with respect to the index n of the outer sum. Thus, (6.4.2) converges quite rapidly, with only 14 terms needed to determine γ up to an error of order 10^{-14}. If we regard (6.5.1), or (6.4.2), as a single sum, i.e., each partial sum contains only one additional term from the inner sum, then the computations take much longer.

Ramanujan's series for γ converges much more rapidly than the standard series definition for γ, namely,

$$\gamma = \lim_{n\to\infty} C_n, \qquad C_n := \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right). \tag{6.5.2}$$

To compare the use of (6.5.2) with that of (6.5.1), which we used in computing the previous table, we list the first 14 values of $C_n - \gamma$ in the following table.

n	$C_n - \gamma$	n	$C_n - \gamma$
1	0.42278	8	0.061200
2	0.22964	9	0.054528
3	0.15751	10	0.049167
4	0.11982	11	0.044766
5	0.09668	12	0.041088
6	0.081025	13	0.037969
7	0.069731	14	0.035289

For several years, the most effective algorithm for computing γ has been that of R.P. Brent and E.M. McMillan [77]. The current world record, at the writing of this book, for calculating the digits of

$$\gamma = 0.5772156649015328606065120900824024310421593359399 2\ldots$$

is held by Alexander J. Yee and R. Chan [320], who calculated 29,844,489,545 digits.

Another representation for γ can be found in Entry 44 of Chap. 12 in Ramanujan's second notebook [268], [38, p. 167]. Asymptotic expansions for γ are located in Corollaries 1 and 2 in Sect. 9 of Chap. 4 in his second notebook [268], [37, p. 98]. An extension of these results along with an interesting discussion of them has been given by R.P. Brent [75, 76].

7

Problems in Diophantine Approximation

7.1 Introduction

In this chapter, we examine three partial manuscripts on Diophantine approximation found in [269]. All are untitled and in rough form.

The first partial manuscript is on pages 262–265. At the top of page 262 are two appended notes. The first, possibly in the handwriting of G.H. Hardy's former research student, Gertrude Stanley, reads (in part) "Paper a little difficult to understand after the first page." The second, definitely in the handwriting of Hardy, surmises "Odd problem. I don't profess to know whether there's much to it."

On these four pages, Ramanujan considers the problem of finding the maximum value of a certain polynomial when the variable x is a rational number with prescribed denominator. We do not know what motivated Ramanujan to consider this particular problem, and it is natural to ask whether Ramanujan's analysis can be extended to other algebraic numbers. Probably, this is the case, but it appears to be complicated to state and prove a general theorem. Although this problem is outside the scope of contemporary research in Diophantine approximation, because only elementary number theory and elementary calculus are involved, we hope that readers will find Ramanujan's problem and its analysis to be appealing. We have decided that it would be unwise to dwell on every inaccuracy or vague statement in Ramanujan's manuscript. We emphasize that the principal ideas are due to Ramanujan, but that it took considerable effort to interpret and make them precise. The proofs are substantially due S. Kim and the second author [56].

The second manuscript is on pages 266 and 267 of [269]. This short manuscript is more precisely and clearly written. Ramanujan considers the Diophantine approximation of the exponential function $e^{2/a}$, where a is a nonzero integer. Remarkably, he obtains the best possible Diophantine approximation to $e^{2/a}$, a result that was first established in the literature by C.S. Davis [102] in 1978, probably about 60 years after Ramanujan had proved it. Our account of this manuscript is taken from a paper [61] that

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Berndt coauthored with S. Kim and A. Zaharescu. This paper contains further results. In particular, the authors examine how often the convergents to the (simple) continued fraction of e coincide with partial sums of e. Moreover, they prove a conjecture of J. Sondow [292] asserting that only two partial sums of the Maclaurin series for e coincide with partial quotients of the simple continued fraction of e.

We have been unable to provide meaning to the third manuscript, which is on page 343. Its claims are wrong, and so it remains a challenge to determine whether something meaningful can be ascertained.

7.2 The First Manuscript

7.2.1 An Unusual Diophantine Problem

We begin by quoting Ramanujan at the beginning of his manuscript.

> Let us consider the maximum of
>
> $$\epsilon_m(1-\epsilon_m)(1-2\epsilon_m) \tag{7.2.1}$$
>
> when ϵ_m is a positive proper fraction and m and $m\epsilon_m$ are positive integers. Let v_m be the maximum of (7.2.1). If we do not assume that $m\epsilon_m$ is rational, we get that
>
> $$\epsilon_m = \frac{3-\sqrt{3}}{6}, \qquad v_m = \frac{1}{6\sqrt{3}}. \tag{7.2.2}$$

Here, as a *positive proper fraction*, Ramanujan intends ϵ_m to be a rational number (not necessarily in lowest terms) with denominator m. If

$$f(x) := x(1-x)(1-2x) = x - 3x^2 + 2x^3, \tag{7.2.3}$$

then it is easily seen that $x = (3-\sqrt{3})/6$ yields a local maximum of $f(x)$. Ramanujan desires to find the maximum value v_m of (7.2.3) when approximating $(3-\sqrt{3})/6$ by a rational number ϵ_m with denominator equal to m. He then claims that ϵ_m is either

$$g_m(\epsilon) := \frac{m \cdot \left(\dfrac{3-\sqrt{3}}{6}\right) - \epsilon}{m} \qquad \text{or} \qquad g_m(\epsilon - 1) = \frac{m \cdot \left(\dfrac{3-\sqrt{3}}{6}\right) + 1 - \epsilon}{m}. \tag{7.2.4}$$

Here, we can see that ϵ is completely determined by m. We can assume that $0 < \epsilon < 1$, so that the two values in (7.2.4) give the two best rational approximations to $(3-\sqrt{3})/6$ with denominator m. In the first instance of (7.2.4), the approximation is from below, while in the second instance, the approximation is from above. Ramanujan then claims the following.

Proposition 7.2.1. *If*

$$\epsilon_m = \frac{m \cdot \left(\frac{3-\sqrt{3}}{6}\right) - \epsilon}{m}, \quad \textit{then} \quad v_m = \frac{1}{6\sqrt{3}} - \frac{\epsilon^2}{m^2}\sqrt{3} - 2\frac{\epsilon^3}{m^3}, \tag{7.2.5}$$

and if

$$\epsilon_m = \frac{m \cdot \left(\frac{3-\sqrt{3}}{6}\right) + 1 - \epsilon}{m}, \quad \textit{then} \quad v_m = \frac{1}{6\sqrt{3}} - \frac{(1-\epsilon)^2}{m^2}\sqrt{3} + 2\frac{(1-\epsilon)^3}{m^3}. \tag{7.2.6}$$

Proof. With the use of (7.2.3), both of these calculations are straightforward. □

We note that by replacing ϵ by $\epsilon - 1$ in the value of v_m in (7.2.5), we obtain the value of v_m in (7.2.6).

Proposition 7.2.2.

$$\textit{If} \qquad \epsilon < \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \qquad \text{then } v_m \text{ in (7.2.5) is greater;} \tag{7.2.7}$$

$$\textit{if} \qquad \epsilon > \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \qquad \text{then } v_m \text{ in (7.2.6) is greater;} \tag{7.2.8}$$

and

$$\textit{if} \;\; \epsilon = \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \text{ then the values of } v_m \text{ in (7.2.5) and (7.2.6) are identical.} \tag{7.2.9}$$

Proof. An elementary calculation shows that

$$\frac{1}{6\sqrt{3}} - \frac{\epsilon^2}{m^2}\sqrt{3} - 2\frac{\epsilon^3}{m^3} > \frac{1}{6\sqrt{3}} - \frac{(1-\epsilon)^2}{m^2}\sqrt{3} + 2\frac{(1-\epsilon)^3}{m^3} \tag{7.2.10}$$

if and only if

$$6\epsilon^2 + (2m\sqrt{3} - 6)\epsilon + 2 - m\sqrt{3} < 0. \tag{7.2.11}$$

It is easily checked that the roots of $6\epsilon^2 + (2m\sqrt{3} - 6)\epsilon + 2 - m\sqrt{3} = 0$ are

$$r_1, r_2 := \frac{1}{2} + \frac{-m \pm \sqrt{m^2-1}}{2\sqrt{3}}, \qquad \text{with } r_2 < r_1.$$

Thus, (7.2.10) is true if and only if $r_2 < \epsilon < r_1$. Since the root that we seek is r_1, we see that the statements in Proposition 7.2.2 follow. □

Now, if

$$0 < \epsilon < \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}},$$

then

$$\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}} < m\epsilon_m < m\frac{3-\sqrt{3}}{6} < \frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}.$$

Also, if

$$\frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}} < \epsilon < 1,$$

then

$$\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}} < m\frac{3-\sqrt{3}}{6} < m\epsilon_m < \frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}.$$

Thus, if

$$\epsilon \neq \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}},$$

we conclude that the maximum v_m occurs when

$$\epsilon_m = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right]. \tag{7.2.12}$$

We also note that for those values of m for which

$$\epsilon = \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}}, \tag{7.2.13}$$

by (7.2.9), we can choose either expression from (7.2.4) for ϵ_m. Thus,

$$\epsilon_m = \frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \quad \text{or} \quad \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right). \tag{7.2.14}$$

We remark that by (7.2.4) and (7.2.13), we do not need greatest integer functions in (7.2.14). Hence, we have established the following proposition.

Proposition 7.2.3. *The formula for ϵ_m in (7.2.12) is valid for all values of m, and in the case of (7.2.13), ϵ_m can be determined by the alternative choices in (7.2.14).*

In conclusion, we use (7.2.12) to calculate ϵ_m. We then return to (7.2.3) to determine v_m.

In Table 7.1, we list the values of ϵ_m for each m, $1 \leq m \leq 10$, which were obtained from (7.2.12) or (7.2.14). We also add the corresponding values of ϵ in the table.

Ramanujan next discusses the *minimum order* and *maximum order* of v_m. He does not define these concepts, but in different words we relate what we think he intended.

m	ϵ_m	Value of ϵ	v_m
1	0, 1	$\frac{1}{2} - \frac{1}{2\sqrt{3}}$	0
2	$0, \frac{1}{2}$	$\frac{3-\sqrt{3}}{3}$	0
3	$\frac{1}{3}$	$\frac{3-\sqrt{3}}{2}$	$\frac{2}{27}$
4	$\frac{1}{4}$	$2 - \frac{2\sqrt{3}}{3}$	$\frac{3}{32}$
5	$\frac{1}{5}$	$\frac{3}{2} - \frac{5\sqrt{3}}{6}$	$\frac{12}{125}$
6	$\frac{1}{6}$	$2-\sqrt{3}$	$\frac{20}{6^3}$
7	$\frac{1}{7}, \frac{2}{7}$	$\frac{5}{2} - \frac{7\sqrt{3}}{6}$	$\frac{30}{7^3}$
8	$\frac{1}{4}$	$3 - \frac{4\sqrt{3}}{3}$	$\frac{3}{32}$
9	$\frac{2}{9}$	$\frac{7}{2} - \frac{3\sqrt{3}}{2}$	$\frac{70}{3^6}$
10	$\frac{1}{5}$	$3 - \frac{5\sqrt{3}}{3}$	$\frac{12}{5^3}$

Table 7.1. Table of values for v_m, $1 \le m \le 10$

Proposition 7.2.4. *For all values of* m,

$$v_m \ge \frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}}, \tag{7.2.15}$$

with equality holding when

$$\epsilon = \frac{1}{2} - \frac{m-\sqrt{m^2-1}}{2\sqrt{3}}, \tag{7.2.16}$$

and the corresponding value of ϵ_m *is given by*

$$\epsilon_m = \frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \quad \textit{or} \quad \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right). \tag{7.2.17}$$

Proof. From (7.2.12), we have

$$\frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \le \epsilon_m \le \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right).$$

If the maximum v_m occurs at $\epsilon_m \leq (3-\sqrt{3})/6$, then

$$v_m \geq f\left(\frac{1}{m}\left(\frac{m-1}{2}-\sqrt{\frac{m^2-1}{12}}\right)\right)=\frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}},$$

since $f(x)$ is increasing when $x \leq (3-\sqrt{3})/6$. On the other hand, $f(x)$ is decreasing when $(3-\sqrt{3})/6 \leq x \leq 1$. Thus, if the maximum v_m occurs at $\epsilon_m \geq (3-\sqrt{3})/6$, then

$$v_m \geq f\left(\frac{1}{m}\left(\frac{m+1}{2}-\sqrt{\frac{m^2-1}{12}}\right)\right)=\frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}},$$

which completes the proof. □

The previous proposition gives a lower bound for v_m. The next two propositions give upper bounds, with Proposition 7.2.5 due to Ramanujan; Proposition 7.2.6 was not given by Ramanujan in his partial manuscript.

Proposition 7.2.5. *If* $\epsilon_m = g(\epsilon)$, *then*

$$v_m \leq \frac{m^2-1}{6m^3}\sqrt{\frac{m^2+2}{3}}, \tag{7.2.18}$$

with equality holding above when

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right). \tag{7.2.19}$$

Proof. We first note that

$$m\frac{3-\sqrt{3}}{6}=\frac{m}{2}-\sqrt{\frac{m^2}{12}} \quad \text{and} \quad \frac{m}{2}-\sqrt{\frac{m^2+1}{12}}$$

cannot be integers, whereas

$$\frac{m}{2}-\sqrt{\frac{m^2+2}{12}} \tag{7.2.20}$$

is an integer for $m = 1, 5, 19, \ldots$. Also, it can easily be verified that

$$\left\lceil\frac{m}{2}-\sqrt{\frac{m^2}{12}}\right\rceil=\left\lceil\frac{m}{2}-\sqrt{\frac{m^2+1}{12}}\right\rceil=\left\lceil\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right\rceil \leq \frac{m}{2}-\sqrt{\frac{m^2+2}{12}}.$$

Thus, we obtain

$$v_m \leq f\left(\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right)\right)=\frac{m^2-1}{6m^3}\sqrt{\frac{m^2+2}{3}}.$$

□

Proposition 7.2.6. *If* $\epsilon_m = g(\epsilon - 1)$, *then*

$$v_m \leq \frac{m^2+2}{6m^3}\sqrt{\frac{m^2-4}{3}}, \tag{7.2.21}$$

with equality holding when

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2-4}{12}}\right). \tag{7.2.22}$$

Proof. First, it can be easily verified that for $0 \leq i \leq 3$,

$$\frac{m}{2} - \sqrt{\frac{m^2-i}{12}}$$

does not take any integral values. So, we have

$$\left\lceil \frac{m}{2} - \sqrt{\frac{m^2}{12}} \right\rceil = \left\lceil \frac{m}{2} - \sqrt{\frac{m^2-4}{12}} \right\rceil \geq \frac{m}{2} - \sqrt{\frac{m^2-4}{12}}.$$

Thus, we obtain

$$v_m \leq f\left(\frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2-4}{12}}\right)\right) = \frac{m^2+2}{6m^3}\sqrt{\frac{m^2-4}{3}}.$$

□

This concludes the first section of Ramanujan's partial manuscript.

7.2.2 The Periodicity of v_m

In the second and last section of his draft, Ramanujan considers the periodicity of v_m. To motivate the remainder of our paper, we move his table from the end of the manuscript to the beginning of this section (see Table 7.2).

We observe that there exist sequences of values that are periodic, e.g.,

$$v_5 = v_{10} = v_{15} = v_{20} = v_{25} = v_{30} = v_{35} = v_{40}. \tag{7.2.23}$$

Ramanujan then seeks to determine the maximum value of k such that

$$v_m = v_{2m} = v_{3m} = \cdots = v_{km}. \tag{7.2.24}$$

Theorem 7.2.1. *As in* (7.2.19), *consider only those values of* m *for which*

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right) \tag{7.2.25}$$

$v_1 = 0$	$v_{26} = 0.0955849$
$v_2 = 0$	$v_{27} = 0.0960219$
$v_3 = 0.0740741$	$v_{28} = 0.0962099$
$v_4 = 0.0937500$	$v_{29} = 0.0961909$
$v_5 = 0.0960000$	$v_{30} = 0.0960000$
$v_6 = 0.0925926$	$v_{31} = 0.0958679$
$v_7 = 0.0874436$	$v_{32} = 0.0961304$
$v_8 = 0.0937500$	$v_{33} = 0.0962239$
$v_9 = 0.0960219$	$v_{34} = 0.0961734$
$v_{10} = 0.0960000$	$v_{35} = 0.0960000$
$v_{11} = 0.0946657$	$v_{36} = 0.0960219$
$v_{12} = 0.0937500$	$v_{37} = 0.0961838$
$v_{13} = 0.0955849$	$v_{38} = 0.0962239$
$v_{14} = 0.0962099$	$v_{39} = 0.0961581$
$v_{15} = 0.0960000$	$v_{40} = 0.0960000$
$v_{16} = 0.0952148$	$v_{41} = 0.0961100$
$v_{17} = 0.0952575$	$v_{42} = 0.0962099$
$v_{18} = 0.0960219$	$v_{43} = 0.0962179$
$v_{19} = 0.0962239$	$v_{44} = 0.0961448$
$v_{20} = 0.0960000$	$v_{45} = 0.0960219$
$v_{21} = 0.0954541$	$v_{46} = 0.0961617$
$v_{22} = 0.0957926$	$v_{47} = 0.0962215$
$v_{23} = 0.0961617$	$v_{48} = 0.0962095$
$v_{24} = 0.0962095$	$v_{49} = 0.0961334$
$v_{25} = 0.0960000$	$v_{50} = 0.0960960$

Table 7.2. Table of values for v_m, $1 \leq m \leq 50$

is a rational number. Let k be the maximum value such that (7.2.24) holds. Then

$$k \not> \left[\frac{x}{m}\right] = \sqrt{3m^2+6} - 1, \tag{7.2.26}$$

where x is determined by

$$\frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right)=\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right)=\epsilon_m. \tag{7.2.27}$$

Proof. From (7.2.12), recall that for every m

$$\epsilon_m=\frac{1}{m}\left\lfloor\frac{m+1}{2}-\sqrt{\frac{m^2-1}{12}}\right\rfloor,$$

or in terms of the least integer function,

$$\epsilon_m=\frac{1}{m}\left\lceil\frac{m-1}{2}-\sqrt{\frac{m^2-1}{12}}\right\rceil.$$

With these values in mind, we first examine, for $x>1$, the two functions

$$f_1(x):=\frac{1}{x}\left(\frac{x+1}{2}-\sqrt{\frac{x^2-1}{12}}\right) \quad\text{and}\quad f_2(x):=\frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right).$$

An elementary calculation shows that

$$\begin{aligned} f_1'(x)&=-\frac{1}{2x^2}-\frac{1}{12x^2}\left(\frac{x^2-1}{12}\right)^{-1/2}<0,\\ f_2'(x)&=\frac{1}{2x^2}-\frac{1}{12x^2}\left(\frac{x^2-1}{12}\right)^{-1/2}>0, \end{aligned}$$

provided that $x>2/\sqrt{3}$. Thus, $f_1(x)$ is monotonically decreasing and $f_2(x)$ is monotonically increasing for $x>2/\sqrt{3}$. Also, we see that

$$f_2(x)=\frac{1}{2}-\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}<\frac{3-\sqrt{3}}{6}<f_1(x)=\frac{1}{2}+\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}. \tag{7.2.28}$$

Now we verify (7.2.26). Suppose that we have the sequence of equal values (7.2.24), which, in turn, implies that

$$\epsilon_m=\epsilon_{2m}=\epsilon_{3m}=\cdots=\epsilon_{km}.$$

Since $\epsilon_m=\epsilon_{km}$, by (7.2.12) and (7.2.27),

$$\begin{aligned} \frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right)&=\frac{1}{km}\left\lfloor\frac{km+1}{2}-\sqrt{\frac{k^2m^2-1}{12}}\right\rfloor\\ &\geq\frac{1}{km}\left(\frac{km-1}{2}-\sqrt{\frac{k^2m^2-1}{12}}\right). \end{aligned}$$

Since $f_2(x)$ is monotonically increasing, it follows that $x\geq km$, which proves the first equality in (7.2.26).

Now we solve (7.2.27). Let

$$\alpha = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right).$$

Then, a straightforward calculation shows that

$$x = \frac{3(1-2\alpha) \pm \sqrt{1+12\alpha-12\alpha^2}}{3(1-2\alpha)^2-1}. \tag{7.2.29}$$

Since $\alpha = \frac{1}{2} - \frac{1}{m}\sqrt{\frac{m^2+2}{12}}$, we easily find that

$$\begin{aligned} 1+12\alpha-12\alpha^2 &= 4-12\left(\frac{1}{2}-\alpha\right)^2 = \frac{3m^2-2}{m^2}, \\ 3(1-2\alpha) &= \frac{\sqrt{3m^2+6}}{m}, \\ 3(1-2\alpha)^2-1 &= 2-12\alpha+12\alpha^2 = 12\left(\alpha-\frac{1}{2}\right)^2-1 = \frac{2}{m^2}. \end{aligned}$$

Hence, by (7.2.29), we deduce that

$$\frac{x}{m} = \frac{\sqrt{3m^2+6}+\sqrt{3m^2-2}}{2}.$$

However, by (7.2.25), we see that $(m^2+2)/3$ is a perfect square, which is equivalent to $3m^2+6$ being a perfect square. Thus,

$$k \leq \left[\frac{x}{m}\right] = \sqrt{3m^2+6}-1, \tag{7.2.30}$$

which verifies the second equality in (7.2.26). □

In the next result, Ramanujan removes the restriction on (7.2.25) from Theorem 7.2.1 and claims a formula that is valid for *all* m.

Theorem 7.2.2. *Assume that* x *is chosen so that either*

$$\frac{1}{x}\left(\frac{x+1}{2} - \sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] > \frac{3-\sqrt{3}}{6} \tag{7.2.31}$$

or

$$\frac{1}{x}\left(\frac{x-1}{2} - \sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] < \frac{3-\sqrt{3}}{6}. \tag{7.2.32}$$

Then

$$k = \left[\frac{x}{m}\right]. \tag{7.2.33}$$

Moreover, if $3m^2 + 6$ *is a perfect square, then*

$$k = \sqrt{3m^2+6} - 1. \tag{7.2.34}$$

Proof. Observe that the last statement in Theorem 7.2.2 follows from (7.2.33) and (7.2.30).

In order to prove (7.2.33), we first need to show that $k \le [x/m]$ for arbitrary m. In the case of (7.2.32), we can use the same argument from the proof of (7.2.26). For the case of (7.2.31), if we assume $\epsilon_m = \epsilon_{2m} = \cdots = \epsilon_{km}$, then we have

$$\begin{aligned}\frac{1}{x}\left(\frac{x+1}{2} - \sqrt{\frac{x^2-1}{12}}\right) &= \frac{1}{km}\left[\frac{km+1}{2} - \sqrt{\frac{k^2m^2-1}{12}}\right]\\ &\le \frac{1}{km}\left(\frac{km+1}{2} - \sqrt{\frac{k^2m^2-1}{12}}\right).\end{aligned}$$

Since $f_1(x)$ is monotonically decreasing, we conclude that $km \le x$, or $k \le [x/m]$.

We now show that for every $1 \le t \le [x/m]$, $\epsilon_m = \epsilon_{tm}$, which proves (7.2.33). We first consider those values of m for which (7.2.31) holds. Since all the rational numbers with denominator tm include the rational numbers with denominator m, we have $v_{tm} \ge v_m$. Since $\epsilon_m > (3-\sqrt{3})/6$ and the function $f(x) = x(1-x)(1-2x)$ is decreasing on the interval $[(3-\sqrt{3})/6, 1]$, we have $\epsilon_{tm} \le \epsilon_m$. On the other hand, since $f_1(x)$ is decreasing, by (7.2.31),

$$\epsilon_m = \frac{t}{tm}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \le \frac{1}{tm}\left(\frac{tm+1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right).$$

Thus,

$$t\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \le \left[\frac{tm+1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right],$$

which implies that $\epsilon_m \le \epsilon_{tm}$, upon dividing both sides above by tm. Hence, the inequalities $\epsilon_m \ge \epsilon_{tm}$ and $\epsilon_m \le \epsilon_{tm}$ imply that $\epsilon_m = \epsilon_{tm}$ for all $1 \le t \le [x/m]$.

For those values of m that satisfy (7.2.32), we apply a similar argument. Since $v_{tm} \ge v_m$ and the function $f(x)$ is increasing on the interval $[0, (3-\sqrt{3})/6]$, we have $\epsilon_m \le \epsilon_{tm}$. Since $f_2(x)$ is increasing, by (7.2.32),

$$\epsilon_m = \frac{t}{tm}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \ge \frac{1}{tm}\left(\frac{tm-1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right).$$

Thus,

$$t\left[\frac{m+1}{2}-\sqrt{\frac{m^2-1}{12}}\right] \geq \left[\frac{tm-1}{2}-\sqrt{\frac{t^2m^2-1}{12}}\right],$$

which implies that $\epsilon_{tm} \leq \epsilon_m$, upon dividing both sides above by tm. Thus, since we also had observed that $\epsilon_m \leq \epsilon_{tm}$, we conclude that $\epsilon_m = \epsilon_{tm}$, which completes the proof of (7.2.33). □

In summary, if $3m^2+6$ is a perfect square, then we use (7.2.34) to calculate the length k of the period. If $3m^2+6$ is not a perfect square, then we use (7.2.33), with x defined by (7.2.31) or (7.2.32), to calculate the period length k.

If $m=1$, then by (7.2.34), $k=2$. In our initial calculations above, we had observed that $v_1=v_2=0$, but $v_3\neq 0$, and so Ramanujan's periodic assertion is corroborated in this case. Ramanujan then gives seven periodic sequences corresponding to the values $m=5,9,14,19,71,265,989$, with periods $8,5,12,32,122,458,1{,}712$, respectively, namely,

$$\begin{aligned}
&v_5=v_{10}=v_{15}=\cdots=v_{40},\\
&v_9=v_{18}=v_{27}=\cdots=v_{45},\\
&v_{14}=v_{28}=v_{42}=\cdots=v_{168},\\
&v_{19}=v_{38}=v_{57}=\cdots=v_{608},\\
&v_{71}=v_{142}=v_{213}=\cdots=v_{8,662},\\
&v_{265}=v_{530}=v_{795}=\cdots=v_{121,370},\\
&v_{989}=v_{1,978}=v_{2,967}=\cdots=v_{1,693,168}.
\end{aligned}$$

The first, fourth, fifth, sixth, and seventh sequences arise from (7.2.34), but for the second and third, we must use (7.2.33) and (7.2.31) to determine the values $k=5$ and $k=12$, respectively.

It is interesting to examine how often $3m^2+6$ is a perfect square. If we let $3m^2+6=n^2$ or $n^2-3m^2=6$, then $n+m\sqrt{3}$ is an element of $\mathbb{Z}[\sqrt{3}]$ with norm 6. Since $3+\sqrt{3}$ is such an element with positive smallest values of n and m, and $2+\sqrt{3}$ is the fundamental unit of $\mathbb{Z}[\sqrt{3}]$, all the values of n and m generated by $(3+\sqrt{3})(2+\sqrt{3})^r$ with $r\in\mathbb{Z}$ are solutions. In fact, we can also show that they are the only solutions, using the LMM algorithm as described by K. Matthews [221], for example. We remark that the values $m=5,19,71,265,989$ are generated by $(3+\sqrt{3})(2+\sqrt{3})^r$ with $1\leq r\leq 5$.

We complete our discussion of this first manuscript by adding an explanation for those readers who are reading this chapter in conjunction with Ramanujan's original manuscript. In fact, instead of (7.2.27) in Theorem 7.2.1, Ramanujan had written

$$\frac{1}{x}\left(\frac{x+1}{2}-\sqrt{\frac{x^2-1}{12}}\right)=\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right). \tag{7.2.35}$$

Now, the right-hand side of (7.2.35) is

$$\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right)=\frac{1}{2}-\sqrt{\frac{1}{12}+\frac{1}{6m^2}}<\frac{3-\sqrt{3}}{6}, \tag{7.2.36}$$

while the left-hand side of (7.2.35), by (7.2.28), is equal to

$$\frac{1}{2}+\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}=f_1(x)>\frac{3-\sqrt{3}}{6}. \tag{7.2.37}$$

Clearly, (7.2.36) and (7.2.37) are incompatible. This mistake caused confusion for the writer of the first note appended to Ramanujan's manuscript. She (or he) writes, "I don't see where eqn (7.2.26) (the second equality) comes from, e.g., $m = 5$, $k = 8$ does not come from the value of k given $[x/m]$, as x is negative."

7.3 A Manuscript on the Diophantine Approximation of $e^{2/a}$

In this section, we discuss the partial manuscript on pages 266–267 of [269], in which Ramanujan examines the Diophantine approximation of $e^{2/a}$ when a is a nonzero integer. At the top of page 266 is a note, "See Q. 784(ii) in volume. This goes further," which is in G.H. Hardy's handwriting. Question 784 is a problem on the Diophantine approximation submitted by Ramanujan to the *Journal of the Indian Mathematical Society* [261] [267, p. 334]; "volume" evidently refers to Ramanujan's *Collected Papers* [267]. It took more than a decade before A.A. Krishnaswami Aiyangar [203] published a partial solution and T. Vijayaraghavan and G.N. Watson [309] published a complete solution to Question 784. In Question 784, Ramanujan improved upon the classical approximation. But in the partial manuscript on pages 266 and 267, Ramanujan made a further improvement and moreover derived the best possible Diophantine approximation for $e^{2/a}$. As remarked in the introduction, such a theorem was first proved in print by C.S. Davis [102] in 1978, approximately 60 years after Ramanujan discovered it. Of course, Davis was unaware that his theorem was ensconced in Ramanujan's lost notebook. As we indicate in the sequel, Ramanujan's proof is different, and considerably more elementary, than Davis's proof. Thus, Hardy's remark is on the mark. Using methods similar to those of Ramanujan (but of course, without knowledge of Ramanujan's work), B.G. Tasoev [300] established a general result, for which Davis's theorem is a special case. In regard to Ramanujan's original problem, readers might find a letter from S.D. Chowla to S.S. Pillai, written on August 25, 1929, of interest [20, p. 612].

7.3.1 Ramanujan's Claims

Ramanujan established three different, but related, results, which we relate in a moderately more contemporary style. As customary, $[x]$ denotes the greatest integer in x.

Entry 7.3.1 (p. 266). *Let $\epsilon > 0$ be given. If a is any nonzero integer, then there exist infinitely many positive integers N such that*

$$Ne^{2/a} - [Ne^{2/a}] < \frac{(1+\epsilon)\log\log N}{|a|N\log N}. \tag{7.3.1}$$

Moreover, for all sufficiently large positive integers N,

$$Ne^{2/a} - [Ne^{2/a}] > \frac{(1-\epsilon)\log\log N}{|a|N\log N}. \tag{7.3.2}$$

Entry 7.3.1 might be compared with a theorem of P. Bundschuh established in 1971 [84]. If t is a nonzero integer, then there exist positive constants c_1 and infinitely many rational numbers p/q such that

$$\left|e^{1/t} - \frac{p}{q}\right| < c_1 \frac{\log\log q}{q^2\log q};$$

and there exists a positive constant c_2 such that for all rational numbers p/q,

$$\left|e^{1/t} - \frac{p}{q}\right| > c_2 \frac{\log\log q}{q^2\log q}.$$

In his next theorem, Ramanujan considers two cases, – a even and a odd. His result for a even is identical to that for Entry 7.3.1, except that he formulates his conclusion in terms of $1 + [Ne^{2/a}] - Ne^{2/a}$. We therefore state Ramanujan's claim only in the case that a is odd.

Entry 7.3.2 (p. 266). *If a is any odd integer and $\epsilon > 0$ is given, then there exist infinitely many positive integers N such that*

$$1 + [Ne^{2/a}] - Ne^{2/a} < \frac{(1+\epsilon)\log\log N}{4|a|N\log N}. \tag{7.3.3}$$

Furthermore, given $\epsilon > 0$, for all positive integers N sufficiently large,

$$1 + [Ne^{2/a}] - Ne^{2/a} > \frac{(1-\epsilon)\log\log N}{4|a|N\log N}. \tag{7.3.4}$$

It will be seen, from the proofs of these entries below, that the constants multiplying

$$\frac{\log\log N}{N\log N}$$

on the right-hand sides of (7.3.1)–(7.3.4) are optimal.

We now provide a precise statement of Davis's theorem [102, Theorem 2], which readers will immediately see is equivalent to Ramanujan's Entries 7.3.1 and 7.3.2. In his paper, Davis, in fact, proves his theorem only in the special case of e, indicating that the proof of the more general result follows along the same lines. Although both the proofs of Davis and Ramanujan employ continued fractions, they are quite different. Davis uses, for example, integrals, hypergeometric functions, and Tannery's theorem. On the other hand, Ramanujan utilizes only elementary properties of continued fractions.

Theorem 7.3.1. *Let $a = \pm 2/t$, where t is a positive integer, and set*

$$c = \begin{cases} 1/t, & \text{if } t \text{ is even}, \\ 1/(4t), & \text{if } t \text{ is odd}. \end{cases}$$

Then, for each $\epsilon > 0$, the inequality

$$\left| e^a - \frac{p}{q} \right| < (c+\epsilon)\frac{\log\log q}{q^2 \log q}$$

has an infinity of solutions in integers p, q. Furthermore, there exists a number q', depending only on ϵ and t, such that

$$\left| e^a - \frac{p}{q} \right| > (c-\epsilon)\frac{\log\log q}{q^2 \log q}$$

for all integers p, q, with $q \geq q'$.

7.3.2 Proofs of Ramanujan's Claims on Page 266

Proof. We begin with the continued fraction

$$\tanh x = \frac{x}{1}\,+\,\frac{x^2}{3}\,+\,\frac{x^2}{5}\,+\,\frac{x^2}{7}\,+\cdots, \qquad x \in \mathbb{C}, \tag{7.3.5}$$

first established by J.H. Lambert, and rediscovered by Ramanujan, who recorded it in his second notebook [268, Chap. 12, Sect. 18], [38, p. 133, Corollary 3]. Write

$$\tanh x = 1 - \frac{2}{e^{2x}+1}$$

in (7.3.5), solve for $2/(e^{2x}+1)$, take the reciprocal of both sides, and set $x = 1/a$, where a is any nonzero integer. Hence,

$$\frac{1}{2}\left(e^{2/a}+1\right) = \frac{1}{1}\,-\,\frac{1}{a}\,+\,\frac{1}{3a}\,+\,\frac{1}{5a}\,+\,\frac{1}{7a}\,+\cdots. \tag{7.3.6}$$

Now consider the nth approximant u_n/v_n of (7.3.6) [218, pp. 8–9], [38, p. 105, Entry 1], i.e., for $n \geq 3$,

$$\frac{1}{1} - \frac{1}{a} + \frac{1}{3a} + \frac{1}{5a} + \frac{1}{7a} + \cdots + \frac{1}{(2n-3)a} = \frac{u_n}{v_n}.$$

Then, provided that $|a| \geq 2$,

$$u_1 = 1, \quad v_1 = 1; \qquad u_2 = |a|, \quad v_2 = |a-1|. \tag{7.3.7}$$

Also, from standard recurrence relations [218, pp. 8–9],

$$u_{n+1} - u_{n-1} = (2n-1)|a|u_n; \qquad v_{n+1} - v_{n-1} = (2n-1)|a|v_n. \tag{7.3.8}$$

From the second equality in (7.3.8), we can deduce that

$$v_{n+1} \sim 2|a|nv_n \quad \text{and} \quad \log v_n \sim n \log n, \tag{7.3.9}$$

as $n \to \infty$.

Now in general, if we define $v_0 = 1$, then [38, p. 105, Entry 1] [312, p. 18]

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} =: a_1 \frac{u_n}{v_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1} a_1 a_2 \cdots a_k}{v_{k-1} v_k}.$$

If we use the formula above in (7.3.6), we easily find that

$$\frac{1}{2}\left(e^{2/a} + 1\right) = \frac{u_n}{v_n} + (-1)^n \left(\frac{1}{v_n v_{n+1}} - \frac{1}{v_{n+1} v_{n+2}} + \cdots \right). \tag{7.3.10}$$

It follows from (7.3.9) and (7.3.10) that as n tends to ∞,

$$e^{2/a} + 1 - \frac{2u_n}{v_n} \sim \frac{(-1)^n}{|a|nv_n^2}. \tag{7.3.11}$$

We now subdivide our examination of (7.3.11) into two cases. First, suppose that a is even. Then, using the fact that v_1 and v_2 in (7.3.7) are odd, the recurrence relation for v_n in (7.3.8), and induction, we easily find that v_n is odd for all $n \geq 1$. Now choose $N = v_n$. By (7.3.9), we see that $n \sim \log N / \log\log N$, as $N \to \infty$. Hence, by (7.3.11), as $N \to \infty$,

$$N(e^{2/a} + 1) - 2u_n \sim \frac{(-1)^n \log\log N}{|a| N \log N}. \tag{7.3.12}$$

Second, suppose that a is odd. Ramanujan then claims that if n is odd, then v_n is odd, while if n is even, then v_n is even. However, these claims are incorrect. By (7.3.7), (7.3.8), and induction, we find, instead, that

$$v_{3m} \text{ and } v_{3m+1} \text{ are odd}; \quad v_{3m+2} \text{ is even}.$$

Thus, choose $N = v_n$, when $n = 3m$ or $n = 3m + 1$. In these cases, as in (7.3.12), we conclude that

$$N(e^{2/a} + 1) - 2u_n \sim \frac{(-1)^n \log\log N}{|a| N \log N}. \tag{7.3.13}$$

However, if $n = 3m + 2$, we can choose $N = \frac{1}{2}v_{3m+2}$. Hence, in this case,

$$N(e^{2/a} + 1) - u_n \sim \frac{(-1)^m \log\log N}{4|a| N \log N}. \tag{7.3.14}$$

Turning to Ramanujan's claims in Entries 7.3.1 and 7.3.2, from the asymptotic formulas (7.3.12) and (7.3.14), we see that all of Ramanujan's claims in these entries readily follow. This completes the proof. □

7.4 The Third Manuscript

Page 343 in the volume [269] containing Ramanujan's lost notebook is devoted to an unusual kind of approximation to certain algebraic numbers. Ramanujan's claims are surprising, and, indeed they do not appear to be valid. We copy page 343 verbatim below, and afterward we briefly discuss Ramanujan's claims:

ℓ, m, n are any integers including 0.

$$\theta = \sqrt[5]{2}.$$

$$a = \frac{1}{\sqrt[5]{2} - 1}, \qquad b = \frac{\sqrt{5}}{(1 + \sqrt[5]{4})^{5/2}}$$

$$a^m b^n \theta = p_{m,n} + \epsilon_{m,n}$$

where $-\frac{1}{2} < \epsilon_{m,n} < \frac{1}{2}$ and $p_{m,n}$ is an integer. Then

$$\epsilon_{m,n} = O\left(\frac{5^{n/2}}{(\sqrt[5]{4} - 2\sqrt[5]{2}\cos\frac{2\pi s}{5} + 1)^{m/2} (\sqrt[5]{16} + 2\sqrt[5]{4}\cos\frac{4\pi s}{5} + 1)^{5n/4}} \right) \tag{7.4.1}$$

where s is the most unfavorable of the integers 1, 2, 3, 4.

$$\theta = \sqrt[7]{2}$$

$$a = \frac{1}{\sqrt[7]{2} - 1}, \qquad b = \frac{7}{(\sqrt[7]{8} - 1)^7}, \qquad c = \frac{\sqrt[7]{2} + 1}{\sqrt[7]{4} - \sqrt[7]{2} + 1},$$

$$a^\ell b^m c^n \theta = p_{\ell,m,n} + \epsilon_{\ell,m,n}$$

$$\epsilon_{\ell,m,n} = O\left(\frac{7^m(\sqrt[7]{4} + 2\sqrt[7]{2}\cos\frac{2\pi s}{5} + 1)^{2n}}{(\sqrt[7]{64} - 2\sqrt[7]{8}\cos\frac{2\pi s}{7} + 1)^{\ell/2}} \right.$$
$$\left. \times \frac{1}{(\sqrt[7]{64} - 2\sqrt[7]{8}\cos\frac{6\pi s}{7} + 1)^{7m/2}(\sqrt[7]{64} + 2\sqrt[7]{8}\cos\frac{6\pi s}{7} + 1)^{n/2}}\right), \tag{7.4.2}$$

where s is the most unfavorable of the integers 1, 2, 3, 4, 5, 6.

We do not know for certain what Ramanujan meant by the term "unfavorable." We think that Ramanujan was indicating that we should choose that value of s that makes the displayed error term the largest. It is unclear why Ramanujan listed $s = 1, 2, 3, 4$ below (7.4.1) instead of just writing $s = 1, 2$, because $\cos\frac{2\pi s}{5} = \cos\frac{8\pi s}{5}$ and $\cos\frac{4\pi s}{5} = \cos\frac{6\pi s}{5}$. Of course, a similar remark holds for the corresponding phrase below (7.4.2). It is also unclear what roles θ play in Ramanujan's thinking.

In order for Ramanujan's claims to have some validity, the numbers $a^m b^n \theta$ and $a^\ell b^m c^n \theta$ would need to become close to integers as ℓ, m, and n become large. It would be astounding if such were the case. Table 7.3 provides some calculations of $p_{m,n}$, $\epsilon_{m,n}$, and the error terms for $s = 1, 2$. We first notice that with increasing m and n, the remainders $\epsilon_{m,n}$ do not appear to be tending to 0, but, as we might expect, appear to be randomly distributing themselves in the interval $[-\frac{1}{2}, \frac{1}{2}]$. Also, note that if we set $m = 0$ and choose $s = 1$, then the error terms in these apparently "unfavorable" instances actually tend to infinity as n tends to infinity. In other words, in order to obtain a meaningful claim in the case $s = 1$, both m and n would both need to tend to infinity. Thus, Ramanujan's claim is meaningless in these cases. Moreover, if we set $m = 0$, then $p_{0,n} \equiv 0$ and $\epsilon_{0,n} \to 0$. Thus, for another reason, to obtain a meaningful claim, both m and n would need to tend to infinity.

If Ramanujan's assertions were correct, then ℓ, m, and n would need to tend to infinity on very special sequences. However, it is doubtful that such sequences exist.

m, n	$a^m b^n \theta$	$p_{m,n}$	$\epsilon_{m,n}$	Error, $s = 1$	Error, $s = 2$
$1, 0$	7.725	8	−0.27	0.7882	0.4892
$2, 0$	51.951	52	−0.05	0.6213	0.2393
$3, 0$	349.372	349	+0.37	0.4897	0.1171
$4, 0$	2,349.532	2, 350	−0.47	0.3860	0.0573
$5, 0$	15,800.658	15, 801	−0.34	0.3042	0.0280
$6, 0$	106,259.805	106, 260	−0.19		
$7, 0$	714,599.734	714, 600	−0.27		
$8, 0$	4,805,700.336	4, 805, 700	+0.34		
$9, 0$	32,318,449.897	32, 318, 450	−0.10		
$10, 0$	217,342,349.872	217, 342, 350	−0.13		
$0, 1$	0.2729	0	+0.27	4.1813	0.4578
$0, 2$	0.0745	0	+0.07	17.4833	0.2096
$0, 3$	0.0203	0	+0.02	73.1028	0.0960
$0, 4$	0.0055	0	+0.01		
$1, 1$	2.108	2	+0.11	3.2958	0.2240
$2, 2$	3.869	4	−0.13	10.8621	0.0502
$3, 3$	7.100	7	+0.10	35.7988	0.0112
$4, 4$	13.031	13	+0.03	117.9842	0.0025
$5, 5$	23.914	24	−0.09		
$6, 6$	23.887	24	−0.11		
$7, 7$	80.543	81	−0.46		
$8, 8$	147.815	148	−0.18		
$9, 9$	271.274	271	+0.27		
$10, 10$	497.849	498	−0.15		

Table 7.3. Values of $p_{m,n}$ and $\epsilon_{m,n}$

8

Number Theory

8.1 In Anticipation of Sathe and Selberg

In the top portion of page 337 in [269], Ramanujan offers the following entry, which we quote.

Entry 8.1.1 (p. 337). *$\phi(x)$ is the number of numbers (not exceeding x) whose number of prime divisors does not exceed k.*

$$\phi(x) \sim \frac{x}{\log x}\left\{1 + \frac{\log\log x}{1!} + \frac{(\log\log x)^2}{2!} + \cdots + \frac{(\log\log x)^{[k]}}{[k]!}\right\}. \quad (8.1.1)$$

This is true when k is infinite. Is this true when k is a function of x?

At the start, it should be pointed out that $\phi(x)$ is not well-defined, because it is not clear if Ramanujan is counting multiplicities of prime factors or not. We shall assume that he did count multiplicities. If k is bounded, then the asymptotic formulas are identical, but if $k > c\log\log x$ for any positive constant c, then they are not.

As an asymptotic formula, only the last term in (8.1.1), which is the largest, is relevant. In fact, with this interpretation, (8.1.1) needs to be slightly corrected. Assuming that k is a positive integer, we should replace the last term in curly brackets by $(\log\log x)^{k-1}/(k-1)!$. One could also interpret (8.1.1) as an asymptotic series, as Ramanujan did in his first sentence below (8.1.1). In the latter interpretation, for an elementary proof of this asymptotic formula (8.1.1), see the text by G.H. Hardy and E.M. Wright [153, §22.18, pp. 368–370]. Undoubtedly, Ramanujan realized that the last term in (8.1.1) is dominant for $k = o(\log\log x)$. As pointed out by A. Granville [128], (8.1.1) is correct for $k < o(\log\log x)$ and for $(k - \log\log x)/\sqrt{\log\log x} \to \infty$. For an account of recent developments on this asymptotic formula and related results, see G. Tenenbaum's books [302, Chaps. II.5, II.6, III.5], [303, Chaps. II.5, II.6, III.5].

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If we consider only distinct prime factors, then our knowledge of the corresponding asymptotic formula is not as complete. In such a case, see papers by A.J. Hildebrand and Tenenbaum [157] and by S. Kerner [181], as well as Tenenbaum's book [303, Chap. II.6].

The question about uniformity was first settled by L.G. Sathe in a series of papers [275–278], and slightly later and more succinctly by A. Selberg [281], [282, pp. 418–422]. We offer one of Selberg's theorems in the formulation given by G. Tenenbaum [303, p. 298, Theorem II.6.4]. As usual, let $\Omega(n)$ denote the total number of prime factors of n. Let

$$N_k(x) := |\{n \le x : \Omega(n) = k\}|.$$

Theorem 8.1.1. *Let $0 < \delta < 1$. Then, there exist positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that, uniformly for $x \ge 3$, $1 \le k \le (2-\delta)\log\log x$, and $N \ge 0$,*

$$N_k(x) = \frac{x}{\log x}\left\{\sum_{j=0}^{N} \frac{Q_{j,k}(\log\log x)}{\log^j x} + O_\delta\left(\frac{(\log\log x)^k}{k!}R_N(x)\right)\right\},$$

where $Q_{j,k}$ is a polynomial of degree at most $k-1$ and

$$R_N(x) := e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N + 1}{\log x}\right)^{N+1}.$$

Further extensions have been accomplished by, among others, H. Delange [103], J.-L. Nicolas [232], and M. Balazard, Delange, and Nicolas [21]. For a thorough discussion of results on this important and famous problem, see Tenenbaum's book [303, Chap. II.6].

8.2 Dickman's Function

Dickman's function $\rho(u)$ was introduced by K. Dickman in 1930 [106], and is defined as follows. For $0 \le u \le 1$, let $\rho(u) \equiv 1$. For each integer $k \ge 1$, $\rho(u)$ is defined inductively for $k \le u \le k+1$ by

$$\rho(u) = \rho(k) - \int_k^u \rho(v-1)\frac{dv}{v}. \tag{8.2.1}$$

Dickman's function is continuous at $u = 1$ and differentiable for $u > 1$. Differentiating (8.2.1), we readily find that $\rho(u)$ satisfies the differential–difference equation

$$u\rho'(u) + \rho(u-1) = 0, \qquad u > 1. \tag{8.2.2}$$

Dickman's function arises naturally in prime number theory. Let $P^+(n)$ denote the largest prime factor of the positive integer n, and set

$$\Psi(x, y) := |\{n \le x : P^+(n) \le y\}|. \tag{8.2.3}$$

Let $u = \log x / \log y$. Then [302, pp. 365–367], uniformly for $x \geq y \geq 2$, as $x \to \infty$,

$$\Psi(x, y) = x\rho(u) + O\left(\frac{x}{\log y}\right). \tag{8.2.4}$$

In particular, for $\sqrt{x} \leq y \leq x$,

$$\Psi(x, y) \sim x(1 - \log u), \tag{8.2.5}$$

and for $x^{1/3} \leq y \leq \sqrt{x}$,

$$\Psi(x, y) \sim x\left(1 - \log u + \int_2^u \log(v - 1)\frac{dv}{v}\right). \tag{8.2.6}$$

With this background, we now record the entry on the lower portion of page 337 in [269].

Entry 8.2.1 (p. 337). *Let $\phi(x)$ denote the number of numbers of the form*

$$2^{a_2}3^{a_3}5^{a_5}\cdots p^{a_p}, \qquad p \leq x^{\epsilon},$$

not exceeding x. Then, for $\frac{1}{2} \leq \epsilon \leq 1$,

$$\phi(x) \sim x\left\{1 - \int_{\epsilon}^{1} \frac{du_0}{u_0}\right\}; \tag{8.2.7}$$

for $\frac{1}{3} \leq \epsilon \leq \frac{1}{2}$,

$$\phi(x) \sim x\left\{1 - \int_{\epsilon}^{1} \frac{du_0}{u_0} + \int_{\epsilon}^{\frac{1}{2}} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\right\}; \tag{8.2.8}$$

for $\frac{1}{4} \leq \epsilon \leq \frac{1}{3}$,

$$\begin{aligned}\phi(x) \sim x\Bigg\{&1 - \int_{\epsilon}^{1} \frac{du_0}{u_0} + \int_{\epsilon}^{\frac{1}{2}} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\\ &- \int_{\epsilon}^{\frac{1}{3}} \frac{du_2}{u_2} \int_{u_2}^{\frac{1}{2}(1-u_2)} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\Bigg\};\end{aligned} \tag{8.2.9}$$

and for $\frac{1}{5} \leq \epsilon \leq \frac{1}{4}$,

$$\begin{aligned}\phi(x) \sim x\Bigg\{&1 - \int_{\epsilon}^{1} \frac{du_0}{u_0} + \int_{\epsilon}^{\frac{1}{2}} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\\ &- \int_{\epsilon}^{\frac{1}{3}} \frac{du_2}{u_2} \int_{u_2}^{\frac{1}{2}(1-u_2)} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\\ &+ \int_{\epsilon}^{\frac{1}{4}} \frac{du_3}{u_3} \int_{u_3}^{\frac{1}{3}(1-u_3)} \frac{du_2}{u_2} \int_{u_2}^{\frac{1}{2}(1-u_2)} \frac{du_1}{u_1} \int_{u_1}^{1-u_1} \frac{du_0}{u_0}\Bigg\};\end{aligned} \tag{8.2.10}$$

and so on.

In the notation (8.2.3), Ramanujan's function $\phi(x)$ is equal to $\Psi(x, x^\epsilon)$. Dickman [106] and N.G. de Bruijn [82] proved the now famous asymptotic formula

$$\Psi(x, x^{1/u}) \sim x\rho(u), \qquad x \to \infty. \tag{8.2.11}$$

For a lucid account of de Bruijn's contributions to prime number theory and, in particular, to Dickman's function, see P. Moree's papers [224] and [225]. From [303, p. 507, Corollary 5.19 (due to Hildebrand); p. 511, Theorem 5.21 (due to Hildebrand and Tenenbaum)], we have, for any $\epsilon > 0$,

$$\Psi(x, y) = x\rho(u) \exp\left\{ O\left(\frac{\log(u+1)}{\log y} + \frac{u}{\exp\{(\log y)^{3/5-\epsilon}\}} \right) \right\},$$

uniformly for $x \geq 2$ and $(\log x)^{1+\epsilon} \leq y \leq x$. This result contains all previous results on smooth approximations to $\Psi(x, y)$.

We now show that Ramanujan's asymptotic formulas (8.2.7)–(8.2.10) are the first four instances of (8.2.11). Thus, although technically Ramanujan did not define Dickman's function $\rho(u)$, if he had stated a general theorem (which he clearly possessed), he obviously would have needed to define a function equal to or equivalent to $\rho(u)$. We are extremely grateful to Hildebrand for the following analysis, including the heuristic argument near the end of this section.

First, we prove Theorem 8.2.1, which, in fact, is a special case of a result of Tenenbaum [301, Eq. (12)]. Second, after stating Theorem 8.2.1, we show that this theorem is equivalent to Entry 8.2.1 of Ramanujan. Third, upon the conclusion of our proof, we give a heuristic approach to Theorem 8.2.1, and we conjecture that this is the method used by Ramanujan to deduce Entry 8.2.1.

Theorem 8.2.1. *Define, for $u \geq 0$,*

$$I_0(u) := 1, \qquad I_k(u) := \int_{\substack{t_1,\dots,t_k \geq 1 \\ t_1+\cdots+t_k \leq u}} \cdots \int \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k}, \qquad k \geq 1. \tag{8.2.12}$$

Then, for $u \geq 0$,

$$\rho(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} I_k(u). \tag{8.2.13}$$

The series on the right-hand side of (8.2.13) is finite, since if $k > u$, then $I_k(u) = 0$, for the conditions $t_1, \dots, t_k \geq 1$ and $t_1 + \cdots + t_k \leq u$ are vacuous in this case.

If we make the changes of variable $\epsilon = 1/u$ and $u_j = \epsilon t_j = t_j/u$, $1 \leq j \leq k$, and use the symmetry of the integral $I_k(u)$ in the variables $t_1, \dots, t_k$, then, as shown below, we see that the kth term on the right-hand side of (8.2.13) is identical to the kth term in the expressions in curly brackets in (8.2.7)–(8.2.10). To that end, for $\epsilon = 1/u$, $u > 1$, and $k \leq 1/\epsilon$,

$$\begin{aligned}\frac{1}{k!}I_k\left(\frac{1}{\epsilon}\right) &= \frac{1}{k!}\underset{\substack{\epsilon\le u_1,\ldots,u_k\le 1\\ u_1+\cdots+u_k\le 1}}{\int\cdots\int}\frac{du_1\cdots du_k}{u_1\cdots u_k}\\ &= \underset{\substack{\epsilon\le u_1\le\cdots\le u_k\le 1\\ u_1+\cdots+u_k\le 1}}{\int\cdots\int}\frac{du_1\cdots du_k}{u_1\cdots u_k}\\ &= \int_\epsilon^{1/k}\frac{du_1}{u_1}\int_{u_1}^{(1-u_1)/(k-1)}\frac{du_2}{u_2}\cdots\int_{u_{k-1}}^{1-u_{k-1}}\frac{du_k}{u_k}.\end{aligned}$$

The integrals $I_k(u)$ have recently appeared in the study of the "Dickman polylogarithm." See papers by D. Broadhurst [79] and K. Soundararajan [296].

Proof of Theorem 8.2.1. Our proof hinges on the identity

$$I_k'(u) = \frac{k}{u}I_{k-1}(u-1), \qquad k\ge 1, u>k. \tag{8.2.14}$$

Assuming (8.2.14) for the time being, we proceed with the proof of Theorem 8.2.1.

For $0\le u\le 1$, $\rho(u)\equiv 1$, by definition, while the series on the right side of (8.2.13) reduces to $I_0(u)$, which equals 1, by definition. Thus, (8.2.13) holds for $0\le u\le 1$.

Next, observe that the right-hand side of (8.2.13) is a continuous function of u. Thus, to show that it is equal to $\rho(u)$ for $u\ge 0$, it suffices to show that for nonintegral values of $u>1$, it satisfies the same differential–difference equation as $\rho(u)$, i.e., (8.2.2). By (8.2.14) and the fact that $I_k(u)=0$ for $k\ge u$, we find that for $u>1$ and u nonintegral,

$$\begin{aligned}\frac{d}{du}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}I_k(u)\right) &= \frac{d}{du}\left(1+\sum_{1\le k<u}\frac{(-1)^k}{k!}I_k(u)\right)\\ &= \sum_{1\le k<u}\frac{(-1)^k}{k!}I_k'(u)\\ &= \sum_{1\le k<u}\frac{(-1)^k}{k!}\frac{k}{u}I_{k-1}(u-1)\\ &= \frac{1}{u}\sum_{0\le k<u-1}\frac{(-1)^{k+1}}{k!}I_k(u-1)\\ &= -\frac{1}{u}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}I_k(u-1),\end{aligned}$$

and so we have shown that the right-hand side of (8.2.13) satisfies (8.2.2), which was our goal.

It remains to prove (8.2.14). For $k = 1$,

$$I_1(u) = \int_1^u \frac{dt}{t}.$$

Thus,

$$I_1'(u) = \frac{1}{u} = \frac{1}{u} I_0(u-1), \qquad \text{for } u > 1.$$

Thus, (8.2.14) is valid for $k = 1$.

For $k \geq 2$, write

$$\begin{aligned}
I_k'(u) &:= \frac{d}{du} \underset{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k \leq u}}{\int \cdots \int} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k} \\
&= \frac{d}{du} \int_1^u \left(\underset{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k = v}}{\int \cdots \int} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k} \right) dv \\
&= \underset{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k = u}}{\int \cdots \int} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k} \\
&=: \mathcal{I}_{k-1}(u), \qquad (8.2.15)
\end{aligned}$$

where $\mathcal{I}_{k-1}$ is a $(k-1)$-dimensional integral, namely, the k-fold convolution of the function

$$f(t) = \begin{cases} 1/t, & t \geq 1, \\ 0, & t < 1, \end{cases}$$

with itself.

Recall from (8.2.14) and (8.2.15) that our goal is to express $\mathcal{I}_{k-1}(u)$ in terms of $\mathcal{I}_{k-1}(u-1)$. Since in the integral definition of $\mathcal{I}_{k-1}(u)$ in (8.2.15), the sum $t_1 + \cdots + t_k$ is equal to u, multiplying the integrand by $(t_1 + \cdots + t_k)/u$ does not change the value of the integral. Hence,

$$\begin{aligned}
\mathcal{I}_{k-1}(u) &= \underset{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k = u}}{\int \cdots \int} \frac{t_1 + \cdots + t_k}{u} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k} \\
&= k \underset{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k = u}}{\int \cdots \int} \frac{t_k}{u} \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k}
\end{aligned}$$

$$= k \underset{\substack{t_1,\ldots,t_{k-1}\geq 1 \\ t_1+\cdots+t_{k-1}\leq u-1}}{\int\cdots\int} \frac{1}{u}\frac{dt_1\cdots dt_{k-1}}{t_1\cdots t_{k-1}}$$

$$= \frac{k}{u} I_{k-1}(u-1).$$

By (8.2.15), we see that (8.2.14) has been proved, which was our intent. □

We now offer a heuristic argument for (8.2.13). Perhaps this was the argument that was used by Ramanujan. Let $x > 0$ and $u \geq 1$. Set

$$\begin{aligned}\mathcal{P} &= \left\{p : p \text{ prime}, \ x^{1/u} < p \leq x\right\}, \\ A &= \{n \in \mathbb{N} : n \leq x\}, \\ A_p &= \{n \in \mathbb{N} : n \leq x, \ p \mid n\}.\end{aligned}$$

Then, by the inclusion–exclusion principle,

$$\begin{aligned}\Psi(x, x^{1/u}) &= |A| - \Big|\bigcup_{p\in\mathcal{P}} A_p\Big| \\ &= |A| - \sum_{p\in\mathcal{P}} |A_p| + \sum_{\substack{p_1<p_2 \\ p_j\in\mathcal{P}}} |A_{p_1} \cap A_{p_2}| \\ &\quad - \sum_{\substack{p_1<p_2<p_3 \\ p_j\in\mathcal{P}}} |A_{p_1} \cap A_{p_2} \cap A_{p_3}| + \cdots \\ &= S_0 - S_1 + S_2 - S_3 + \cdots ,\end{aligned}$$

say. We now show that the kth term, $(-1)^k S_k$ above, can be approximated by the kth term on the right-hand side of (8.2.13), multiplied by the factor x.

For $k = 0$,

$$S_0 = |A| = [x],$$

which is our claim in this case. For $k = 1$, using a familiar estimate [302, p. 16] for the sum of the reciprocals of primes and making the change of variable $y = x^{t/u}$ below, we find that

$$\begin{aligned}S_1 &= \sum_{p\in\mathcal{P}} |A_p| = \sum_{x^{1/u}<p\leq x} \left[\frac{x}{p}\right] \\ &\approx x \sum_{x^{1/u}<p\leq x} \frac{1}{p} \approx x \int_{x^{1/u}}^{x} \frac{dy}{y\log y} = x\int_1^u \frac{dt}{t} = xI_1(u).\end{aligned}$$

For $k \geq 2$, we first note that

$$|A_{p_1} \cap \cdots \cap A_{p_k}| = \begin{cases} \left[\dfrac{x}{p_1\cdots p_k}\right], & \text{if } p_1 < \cdots < p_k \text{ and } p_1\cdots p_k \leq x, \\ 0, & \text{if } p_1 < \cdots < p_k \text{ and } p_1\cdots p_k > x. \end{cases}$$

Summing over $p_1, \ldots, p_k$ and later setting $y_j = x^{t_j/u}$, $1 \le j \le k$, we find that

$$\begin{aligned}
\frac{S_k}{x} &\approx \sum_{\substack{p_1 < \cdots < p_k \\ p_j \in \mathcal{P} \\ p_1 \cdots p_k \le x}} \cdots \sum \frac{1}{p_1 \cdots p_k} \\
&\approx \int_{\substack{x^{1/u} < y_1 < \cdots < y_k \le x \\ y_1 \cdots y_k \le x}} \cdots \int \frac{dy_1 \cdots dy_k}{y_1 \log y_1 \cdots y_k \log y_k} \\
&= \int_{\substack{1 < t_1 < \cdots < t_k \le u \\ t_1 + \cdots + t_k \le u}} \cdots \int \frac{dt_1 \cdots dt_k}{t_1 \cdots t_k} \\
&= \frac{1}{k!} I_k(u),
\end{aligned}$$

where $I_k(u)$ is defined in (8.2.12). Thus, up to the multiplicative factor $(-1)^k$, this is the kth term in the series (8.2.13). This completes our heuristic argument, which, with effort, can be made rigorous.

R.C. Vaughan has informed us that $\rho(s) = (s+1)(F(s+1) - f(s+1))$, where $F(s)$ and $f(s)$ are familiar sieving functions [129, p. 113], and that, moreover, G. Greaves [129, p. 220] had shown that $F(s)$ and $f(s)$ can be represented by integrals similar to those in Theorem 8.2.1 below. Greaves further remarks [129, p. 220] that these representations for $F(s)$ and $f(s)$ are implicit in the work of E. Bombieri [73] and H. Siebert [287].

A representation of $\rho(u)$ similar to that in Theorem 8.2.1 was developed by S. Chowla and T. Vijayaraghavan [97, p. 34, Eq. (5)], [96, pp. 682–688] and can be found in Moree's dissertation [223, p. 30, formula (10)]. Their formula was (slightly) corrected and simplified 2 years later by A.A. Buchstab [83]. Evidently, without any knowledge of its connection with prime number theory, W. Gontcharoff [125] independently discovered an integral representation for Dickman's function in his study of the largest cycle in a random permutation. Another representation for $\rho(u)$ as a sum of multiple integrals of the sort appearing in (8.2.7)–(8.2.10) was posed as a problem by H.G. Diamond and F.S. Wheeler [105]. An extension of (8.2.4) has been made by J.-H. Evertse, Moree, C.L. Stewart, and R. Tijdeman [117] to $\psi_{K,T}(x, y)$, which is defined to be the number of ideals in a number field K of norm $\le X$ composed of prime ideals that lie outside a given finite set of prime ideals T and that have norm $\le Y$. Excellent sources for information on the Dickman function and its prominence in prime number theory are Tenenbaum's treatises [302, Chap. III.5], [303, Chap. III.5] and Moree's dissertation [223].

We close this section with a brief mention of analogues of the foregoing work. Consider the number of polynomials over a finite field of degree n, where all their irreducible factors are of degree $\le d$. Then n/d is the analogue of $\log x/\log y$, and one can give an argument in this setting analogous to what we have given above. Readers might consult M. Car's paper [85] for further information on this theory. An analogue of the Dickman function also arises in studying the length of the largest cycle of a random permutation in the symmetric group of degree n, S_n; see, e.g., a paper by L.A. Shepp and S.P. Lloyd [285]. The Dickman function also occurs in the theory of the Poisson–Dirichlet distribution [15].

8.3 A Formula for $\zeta(\frac{1}{2})$

On page 332 in [269], Ramanujan states two versions of a formula, one of which can be found as Entry 8 of Chap.15 in Ramanujan's second notebook [268], [38, p. 314]. Although the formula may be regarded as a representation for $\zeta(\frac{1}{2})$, it also can be viewed as in identity for an infinite sum of theta functions. After stating the first version, we give a brief survey of the activity generated by the proof of R.J. Evans and the second author [53] in their examination of all the theorems found in Chap. 15 of Ramanujan's second notebook. Then we offer Ramanujan's elegant reformulation of the formula, which had been missed by all other authors, except S. Wigert [316, p. 9], who in 1925 proved, in fact, a more general formula that includes Entry 8.3.2 below as a special case. Of course, he had no knowledge that a special case of his discovery can be found in Ramanujan's second notebook or lost notebook. In another paper [317], he generalized his work even further.

Entry 8.3.1 (p. 332). *Let α and β be positive numbers such that $\alpha\beta = 4\pi^3$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{e^{n^2\alpha} - 1} = \frac{\pi^2}{6\alpha} + \frac{1}{4} + \frac{\sqrt{\beta}}{4\pi}\left\{\zeta\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{\cos(\sqrt{n\beta}) - \sin(\sqrt{n\beta}) - e^{-\sqrt{n\beta}}}{\sqrt{n}(\cosh(\sqrt{n\beta}) - \cos(\sqrt{n\beta}))}\right\}. \tag{8.3.1}$$

The factor $\sqrt{n}$ in the denominator on the right-hand side of (8.3.1) is missing in the formulation in the lost notebook.

D. Klusch [182] published a different version of (8.3.1). However, S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [170, 173] pointed out that a mistake in the proof vitiated the formulation given by Klusch. M. Katsurada [178] generalized (8.3.1) in two different ways. In the first, he extended Ramanujan's result by considering a class of generalized Hurwitz zeta functions that are hypergeometric in nature, and by deriving a generalized formula involving the values of these functions at $\frac{1}{2}$. Second, he extended (8.3.1) by establishing a corresponding formula for the Lerch zeta function. Kanemitsu, Tanigawa, and Yoshimoto in [169] and [170] proved formulas in which $\zeta(\frac{1}{2})$ is replaced by the Riemann zeta function at any rational argument. Wigert [316], in fact, had established a formula for $\zeta(\frac{1}{k})$ for any positive even integer

k. Kanemitsu, Tanigawa, and Yoshimoto [173] next established formulas in which $\zeta(\frac{1}{2})$ is replaced by a multiple Hurwitz zeta function evaluated at rational arguments in $(0,1)$. S. Egami [114] proved an analogue of (8.3.1) for Dirichlet L-functions $L(\frac{1}{2},\chi)$. Kanemitsu, Tanigawa, and Yoshimoto [174] extended Egami's result by establishing a formula for $L(\frac{a}{b},\chi)$, where a/b is rational with a odd and b even. They also provided two numerical examples showing how the rapidly convergent series appearing in their formulas can be used to accurately calculate L-series at $\frac{1}{2}$ and $\frac{1}{4}$.

We now provide Ramanujan's second formulation.

Entry 8.3.2 (p. 332). *Let α and β be positive numbers such that $\alpha\beta = 4\pi^3$. If $\phi(n)$, $n \geq 1$, and $\psi(n)$, $n \geq 1$, are defined by*

$$\sum_{j=1}^{\infty} \frac{x^{j^2}}{1-x^{j^2}} = \sum_{n=1}^{\infty} \phi(n) x^n \tag{8.3.2}$$

and

$$\sum_{j=1}^{\infty} \frac{jx^{j^2}}{1-x^{j^2}} = \sum_{n=1}^{\infty} \psi(n) x^n, \tag{8.3.3}$$

then

$$\sum_{n=1}^{\infty} \phi(n) e^{-n\alpha} = \frac{\pi^2}{6\alpha} + \frac{1}{4} + \frac{\sqrt{\beta}}{2\pi}\left\{\frac{1}{2}\zeta\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{\psi(n)}{\sqrt{n}} e^{-\sqrt{n\beta}}\left(\cos(\sqrt{n\beta}) - \sin(\sqrt{n\beta})\right)\right\}. \tag{8.3.4}$$

The term $\frac{1}{4}$ on the right-hand side of (8.3.4) was inadvertently omitted by Ramanujan in [269].

It is not obvious that (8.3.1) and (8.3.4) are different versions of each other, and so we now demonstrate this. First,

$$\sum_{j=1}^{\infty} \frac{x^{j^2}}{1-x^{j^2}} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x^{j^2 k} = \sum_{n=1}^{\infty}\left(\sum_{j^2|n} 1\right) x^n.$$

Hence, from (8.3.2),

$$\phi(n) = \sum_{j^2|n} 1.$$

Similarly,

$$\sum_{j=1}^{\infty} \frac{jx^{j^2}}{1-x^{j^2}} = \sum_{n=1}^{\infty}\left(\sum_{j^2|n} j\right) x^n,$$

i.e., from (8.3.3),

$$\psi(n) = \sum_{j^2|n} j.$$

Second,

$$\begin{aligned} S :&= \sum_{n=1}^{\infty} \frac{\psi(n)}{\sqrt{n}} e^{-\sqrt{n\beta}} \left(\cos(\sqrt{n\beta}) - \sin(\sqrt{n\beta})\right) \\ &= \sum_{n=1}^{\infty} \sum_{j^2|n} j \frac{1}{\sqrt{n}} e^{-\sqrt{n\beta}} \left(\cos(\sqrt{n\beta}) - \sin(\sqrt{n\beta})\right) \\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sum_{j=1}^{\infty} e^{-j\sqrt{k\beta}} \left(\cos(j\sqrt{k\beta}) - \sin(j\sqrt{k\beta})\right). \end{aligned} \tag{8.3.5}$$

Elementary calculations give

$$\sum_{j=1}^{\infty} e^{-jx} \cos(jy) = \frac{e^{-x}(\cos y - e^{-x})}{1 - 2e^{-x}\cos y + e^{-2x}}$$

and

$$\sum_{j=1}^{\infty} e^{-jx} \sin(jy) = \frac{e^{-x}\sin y}{1 - 2e^{-x}\cos y + e^{-2x}}.$$

Using these last two identities in (8.3.5), we find that

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left\{ \frac{e^{-\sqrt{k\beta}}\left(\cos(\sqrt{k\beta}) - e^{-\sqrt{k\beta}}\right) - e^{-\sqrt{k\beta}}\sin(\sqrt{k\beta})}{1 - 2e^{-\sqrt{k\beta}}\cos(\sqrt{k\beta}) + e^{-2\sqrt{k\beta}}} \right\} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left\{ \frac{\cos(\sqrt{k\beta}) - \sin(\sqrt{k\beta}) - e^{-2\sqrt{k\beta}}}{\cosh(\sqrt{k\beta}) - \cos(\sqrt{k\beta})} \right\}. \end{aligned} \tag{8.3.6}$$

If we now use (8.3.6) in (8.3.5), and then substitute this in (8.3.4), after a slight amount of rearrangement, we obtain (8.3.1). This then completes the proof that Ramanujan's elegant formulation (8.3.4) is equivalent to the identity in (8.3.1).

8.4 Sums of Powers

The contents of this section first appeared in a paper by the second author and D. Schultz [67]. On a page published with Ramanujan's lost notebook [269, p. 338], Ramanujan records the following seven equalities.

Entry 8.4.1 (p. 338).

$$x + (x-1) + (x-2) + \cdots = \frac{1}{2}x^2 + \frac{1}{2}x + \begin{cases} +\dfrac{1}{8}, \\ +0, \end{cases}$$

$$x^2 + (x-1)^2 + (x-2)^2 + \cdots = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x + \begin{cases} +\dfrac{1}{36\sqrt{3}}, \\ -\dfrac{1}{36\sqrt{3}}, \end{cases}$$

$$x^3 + (x-1)^3 + (x-2)^3 + \cdots = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \begin{cases} +0, \\ -\dfrac{1}{64}, \end{cases}$$

$$x^4 + (x-1)^4 + (x-2)^4 + \cdots = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x + \begin{cases} +\dfrac{1}{900}\sqrt{15 + 4\sqrt{\tfrac{6}{5}}}, \\ -\dfrac{1}{900}\sqrt{15 + 4\sqrt{\tfrac{6}{5}}}, \end{cases}$$

$$x^5 + (x-1)^5 + (x-2)^5 + \cdots = \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 - \frac{1}{12}x^2 + \begin{cases} +\dfrac{1}{128}, \\ +0, \end{cases}$$

$$x^6 + (x-1)^6 + (x-2)^6 + \cdots = \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 - \frac{1}{6}x^3 + \frac{1}{42}x$$
$$+ \begin{cases} + \\ - \end{cases} \frac{1}{2{,}352}\sqrt{(29\tfrac{2}{3} + 11\sqrt{2})\sqrt[3]{21 - 7\sqrt{2}} + (29\tfrac{2}{3} - 11\sqrt{2})\sqrt[3]{21 + 7\sqrt{2}} - 11},$$

$$x^7 + (x-1)^7 + (x-2)^7 + \cdots = \frac{1}{8}x^8 + \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2 + \begin{cases} +0, \\ -\dfrac{17}{2{,}048}. \end{cases}$$

After these equalities, Hardy has appended a handwritten note, "I'm not clear what this means."

Because Ramanujan had recorded these equations only for his own use, he did not provide interpretations for the left-hand sides or for the brackets on the right-hand sides. However, there is an interpretation that is consistent with the displayed equations (except for a slight error in the equation for sixth powers). We interpret the left-hand side of each equation as the sum of powers of only positive terms, and we interpret the right-hand side as a polynomial $p_n(x)$ plus bounds for the "error term" $R_n(x)$, which we shall explain later. Thus, write

$$s_n(x) := \sum_{i=0}^{\lfloor x \rfloor} (x-i)^n = p_n(x) + R_n(x), \qquad 1 \le n \le 7. \tag{8.4.1}$$

We now interpret and discuss Ramanujan's claims. Ramanujan is claiming values for the extrema of $R_n(x)$ given by the upper and lower values in the brackets. To confirm this, notice first that if x is a positive integer, then

$$p_n(x) = \sum_{j=0}^{x} j^n, \qquad n \ge 1. \tag{8.4.2}$$

Secondly,

$$\begin{aligned} s_n(x+1) &= \sum_{j=0}^{\lfloor x+1 \rfloor} (x+1-j)^n \\ &= \sum_{j=0}^{\lfloor x \rfloor + 1} (x+1-j)^n \\ &= (x+1)^n + \sum_{j=1}^{\lfloor x \rfloor + 1} (x+1-j)^n \\ &= (x+1)^n + s_n(x). \end{aligned} \tag{8.4.3}$$

We also know that the polynomial $p_n(x)$ satisfies the same functional equation for all natural numbers, namely, $p_n(x+1) = (x+1)^n + p_n(x)$. However, since a polynomial is uniquely determined by its values on a finite set of points, $p_n(x+1) = (x+1)^n + p_n(x)$ for all x. This, (8.4.3), and (8.4.1) then imply that $R_n(x+1) = R_n(x)$. So, we can compute its extrema by examining $R_n(x)$ on the interval $[0,1]$. Observe that for $0 \le x < 1$,

$$R_n(x) = x^n - p_n(x), \tag{8.4.4}$$

and so the minimum and maximum can be found from elementary calculus.

We now reformulate Ramanujan's assertions in terms of the familiar Bernoulli polynomials $B_n(x)$, $n \ge 0$, which are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

Recall that the Bernoulli numbers B_n, $n \ge 0$, are defined by $B_n = B_n(0)$, $n \ge 0$. It is easy to show that for $n \ge 1$, $B_{2n+1} = 0$. It is well known that [1, p. 804, Eq. (23.1.4)]

$$\sum_{j=1}^{m} j^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}, \qquad m, n \ge 1. \tag{8.4.5}$$

Furthermore [1, p. 804, Eqs. (23.1.5), (23.1.6), (23.1.8)],

$$B_n'(x) = nB_{n-1}(x), \qquad n \geq 1, \tag{8.4.6}$$

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad n \geq 0, \tag{8.4.7}$$

and

$$B_n(1-x) = (-1)^n B_n(x), \qquad n \geq 0. \tag{8.4.8}$$

In particular, $B_n(1) = (-1)^n B_n$, $n \geq 0$, which, since $B_{2n+1} = 0$, for $n \geq 1$, implies that

$$B_n(1) = B_n, \qquad n \geq 2. \tag{8.4.9}$$

By (8.4.4), (8.4.2), (8.4.5), and (8.4.7), for $n \geq 1$,

$$\begin{aligned} R_n(x) &= x^n - \frac{B_{n+1}(x+1) - B_{n+1}}{n+1} \\ &= x^n - \frac{B_{n+1}(x) - (n+1)x^n - B_{n+1}}{n+1} \\ &= \frac{B_{n+1} - B_{n+1}(x)}{n+1}. \end{aligned} \tag{8.4.10}$$

Hence, by (8.4.6),

$$R_n'(x) = -\frac{B_{n+1}'(x)}{n+1} = -B_n(x), \qquad n \geq 1. \tag{8.4.11}$$

Also, from (8.4.10) and (8.4.9),

$$R_n(0) = 0 \qquad \text{and} \qquad R_n(1) = 0, \qquad n \geq 1. \tag{8.4.12}$$

Lastly, since [1, p. 805, Eq. (23.1.21)],

$$B_n(\tfrac{1}{2}) = -(1 - 2^{1-n})B_n, \qquad n \geq 0, \tag{8.4.13}$$

we can conclude from (8.4.10) that

$$R_n(\tfrac{1}{2}) = \frac{2 - 2^{-n}}{n+1} B_{n+1}, \qquad n \geq 1. \tag{8.4.14}$$

We now establish Ramanujan's claims. We first examine the sums for odd powers n.

For $n = 1$, from (8.4.11),

$$R_1'(x) = -B_1(x) = \tfrac{1}{2} - x,$$

which has the critical point $x = \frac{1}{2}$. Since $B_2 = \frac{1}{6}$, we see from (8.4.14) that $R_1(\frac{1}{2}) = \frac{1}{8}$. Since $R_1(0) = R_1(1) = 0$ by (8.4.12), Ramanujan's first claim is established.

Let $n = 3$. From (8.4.11),

$$R_3'(x) = -B_3(x) = -x(x-1)(x-\tfrac{1}{2}).$$

Since the critical points are $0, 1$, and $\frac{1}{2}$, since $R_3(0) = R_3(1) = 0$ by (8.4.12), and since $R_3 = -\frac{1}{64}$ by (8.4.14), because $B_4 = -\frac{1}{30}$, we verify Ramanujan's assertion for third powers.

Put $n = 5$. Then from (8.4.11),

$$R_5'(x) = -B_5(x) = -x(x-1)(x-\tfrac{1}{2})(x^2 - x - \tfrac{1}{3}),$$

which has the critical points $0, 1, \frac{1}{2}, \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7}{3}}$. Since $B_6 = \frac{1}{42}$, we find from (8.4.14) that

$$R_5\left(\frac{1}{2}\right) = \frac{1}{128} = 0.0078125.$$

Furthermore,

$$R_5\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7}{3}}\right) = 0.00308\ldots.$$

Hence, the maximum and minimum of $R_5(x)$ are $\frac{1}{128}$ and 0, as claimed by Ramanujan.

For $n = 7$, by (8.4.11),

$$R_7'(x) = -B_7(x) = -x(x-1)(x-\tfrac{1}{2})(x^4 - 2x^3 + x + \tfrac{1}{3}),$$

which has the real roots $0, 1, \frac{1}{2}$, and the complex roots

$$\frac{3 \pm \sqrt{3(9 \pm 2i\sqrt{3})}}{6}.$$

Since $B_8 = -\frac{1}{30}$, we find from (8.4.14) that

$$R_7\left(\frac{1}{2}\right) = -\frac{17}{2^{11}},$$

and so the last assertion in Ramanujan's list has been verified.

We now turn to the cases for even powers n. First, if $n = 2$,

$$R_2'(x) = -B_2(x) = -x^2 + x - \tfrac{1}{6},$$

which has the roots $\frac{1}{2}\left(1 \pm \sqrt{\frac{1}{3}}\right)$. Thus, from (8.4.10),

$$R_2\left(\frac{1}{2}\left(1 \pm \sqrt{\frac{1}{3}}\right)\right) = -\frac{1}{3}B_3\left(\frac{1}{2}\left(1 \pm \sqrt{\frac{1}{3}}\right)\right) = \pm\frac{1}{36\sqrt{3}}.$$

Thus, Ramanujan's claim for the sum of squares has been verified.

If $n = 4$, then

$$R_4'(x) = -B_4(x) = -x^4 + 2x^3 - x^2 + \tfrac{1}{30},$$

which has the roots

$$\frac{15 \pm \sqrt{15(15 \pm 2\sqrt{30})}}{30}.$$

Now

$$R_5\left(\frac{15 \mp \sqrt{15(15 - 2\sqrt{30})}}{30}\right) = \pm\frac{1}{900}\sqrt{15 + 4\sqrt{\frac{6}{5}}} = \pm 0.00489164\ldots$$

and

$$R_5\left(\frac{15 \pm \sqrt{15(15 + 2\sqrt{30})}}{30}\right) = \pm\frac{1}{900}\sqrt{15 - 4\sqrt{\frac{6}{5}}} = \pm 0.00362062\ldots.$$

Thus, again Ramanujan's claim is justified.

The calculations for the case $n = 6$ are the most involved, and this is the only case in which the results do not agree with Ramanujan's. Let the roots of the polynomial

$$R_6'(x) = -B_6(x) = -x^6 + 3x^5 - \frac{5x^4}{2} + \frac{x^2}{2} - \frac{1}{42}$$

be denoted by x_i, $1 \leq i \leq 6$. Write the roots of this polynomial as $x_i = \frac{1}{2} + y_i$, where the y_i are roots of

$$y^6 - \frac{5}{4}y^4 + \frac{7}{16}y^2 - \frac{31}{1{,}344}. \tag{8.4.15}$$

The roots of this polynomial can be found by Cardano's formula, and they are

$$12y_i^2 = 5 - 2t_1^2 t_2 - 2t_1 t_2^2, \tag{8.4.16}$$

where

$$t_1 = \sqrt[3]{\frac{11 + 6\sqrt{2}}{7}}, \quad t_2 = \sqrt[3]{\frac{11 - 6\sqrt{2}}{7}}. \tag{8.4.17}$$

Observe that to obtain real roots of (8.4.16), we need to take the real cube roots for t_1 and t_2 in (8.4.17). We find numerically that these real roots, say y_1 and y_2, are $y_1 = 0.2524593\ldots$ and $y_2 = -0.2524593\ldots$.

We would now like to determine the values of $R_6(\frac{1}{2} + y_i)$, $i = 1, 2$. To do this, observe that $R_6(\frac{1}{2} + y)$ is a polynomial of the seventh degree in y, and that if y_i is a root of (8.4.15), we can use this equation to reduce the seventh and sixth powers on y_i to fifth-degree expressions. After elementary algebra, we find that

$$R_6\left(\frac{1}{2}+y_i\right) = \frac{y_i}{4{,}704}\left(93 - 392y_i^2 + 336y_i^4\right).$$

Square both sides, expand, then substitute the values given for y_i^2 from (8.4.16), and finally reduce the exponents on the t_i using the values given in (8.4.17). We thus find that

$$R_6(x_1) = \frac{1}{2{,}352\sqrt{3}}\sqrt{259 - (9-10\sqrt{2})t_1^2t_2 - (9+10\sqrt{2})t_1t_2^2} = 0.0037236\ldots,$$

and $R_6(x_2)$ is the negative of this. An equivalent expression, more in line with that given by Ramanujan, is

$$\pm\frac{1}{2{,}352\sqrt{21}}\sqrt{1{,}813 - (47+39\sqrt{2})\sqrt[3]{21-7\sqrt{2}} - (47-39\sqrt{2})\sqrt[3]{21+7\sqrt{2}}},$$

which is not equal to the expression given by him. It therefore appears that Ramanujan made an error in his calculations, which in view of the computational difficulties above is not surprising. The authors had the advantage of being able to use Mathematica.

In examining the polynomials $B_3(x)$, $B_5(x)$, and $B_7(x)$ above, we note that 0, 1, and $\frac{1}{2}$ are trivial zeros of each, and in general, it is easy to see (e.g., from their Fourier expansions) that these three points are trivial zeros for $B_{2n+1}(x)$, $n \geq 1$. Moreover, J. Lense [212] has shown that these are the only zeros of $B_{2n+1}(x)$, $n \geq 1$, in $[0,1]$. On the basis of four examples above, we might conjecture that $R_{4n+1}(\frac{1}{2})$, $n \geq 0$, yields the maximum value of $R_n(x)$ on $(0,1)$ and that $R_{4n+3}(\frac{1}{2})$, $n \geq 0$, provides the minimum value of $R_n(x)$ on $(0,1)$. Indeed, this is true and was evidently first established by D.H. Lehmer [211].

8.5 Euler's Diophantine Equation $a^3 + b^3 = c^3 + d^3$

On page 341 in his lost notebook [269], Ramanujan offers a truly remarkable method for finding an infinite family of solutions to Euler's diophantine equation $a^3 + b^3 = c^3 + d^3$.

Entry 8.5.1 (p. 341). *If*

$$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = \sum_{n=0}^{\infty} a_n x^n,$$
$$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = \sum_{n=0}^{\infty} b_n x^n,$$

and

$$\frac{2+8x-10x^2}{1-82x-82x^2+x^3}=\sum_{n=0}^{\infty} c_n x^n,$$

then

$$a_n^3+b_n^3=c_n^3+(-1)^n. \tag{8.5.1}$$

This is another of those many results of Ramanujan for which one wonders, "How did he ever think of this?" M.D. Hirschhorn has devoted four papers [141, 158–160] (the former with J.H. Han) to examining Entry 8.5.1. In the remainder of this section, we offer Hirschhorn's approach from [158]. At the close of Sect. 8.5, we offer a few further comments on the work of Ramanujan and Hirschhorn.

First Proof of Entry 8.5.1. For brevity, set

$$A_1=\frac{64+8\sqrt{85}}{85}, \qquad B_1=\frac{64-8\sqrt{85}}{85}, \qquad C_1=\frac{43}{85}, \tag{8.5.2}$$

$$A_2=\frac{77+7\sqrt{85}}{85}, \qquad B_2=\frac{77-7\sqrt{85}}{85}, \qquad C_2=-\frac{16}{85}, \tag{8.5.3}$$

$$A_3=\frac{93+9\sqrt{85}}{85}, \qquad B_3=\frac{93-9\sqrt{85}}{85}, \qquad C_3=\frac{16}{85}. \tag{8.5.4}$$

With the use of partial fractions, it is a straightforward, albeit somewhat tedious, task to show that

$$\begin{aligned} a_n &= A_1\alpha^n + A_2\beta^n + C_1(-1)^n,\\ b_n &= A_2\alpha^n + B_2\beta^n + C_2(-1)^n,\\ c_n &= A_3\alpha^n + B_3\beta^n + C_3(-1)^n, \end{aligned}$$

where

$$\alpha=\frac{83+9\sqrt{85}}{2} \qquad \text{and} \qquad \beta=\frac{83-9\sqrt{85}}{2}.$$

It follows that

$$\begin{aligned} a_n^3 = \frac{1}{85^3}\Big\{&(1{,}306{,}624+141{,}824\sqrt{85})\alpha^{3n}+(1{,}306{,}624-141{,}824\sqrt{85})\beta^{3n}\\ &-(1{,}230{,}144+132{,}096\sqrt{85})(-\alpha^2)^n-(1{,}230{,}144-132{,}096\sqrt{85})(-\beta^2)^n\\ &+(96{,}960+12{,}120\sqrt{85})\alpha^n+(96{,}960-12{,}120\sqrt{85})\beta^n+267{,}245(-1)^n\Big\},\\ b_n^3 = \frac{1}{85^3}\Big\{&(1{,}418{,}648+153{,}664\sqrt{85})\alpha^{3n}+(1{,}418{,}648-153{,}664\sqrt{85})\beta^{3n}\\ &+(484{,}512+51{,}744\sqrt{85}(-\alpha^2)^n+(484{,}512-51{,}744\sqrt{85})(-\beta^2)^n\\ &+(466{,}620+42{,}420\sqrt{85})\alpha^n+(466{,}620-42{,}420\sqrt{85})\beta^n+173{,}440(-1)^n\Big\}, \end{aligned}$$

$$
\begin{aligned}
c_n^3 = \frac{1}{85^3}\Big\{&(2{,}725{,}272+295{,}488\sqrt{85})\alpha^{3n}+(2{,}725{,}272-295{,}488\sqrt{85})\beta^{3n}\\
&-(745{,}632+80{,}352\sqrt{85})(-\alpha^2)^n-(745{,}632-80{,}352\sqrt{85})(-\beta^2)^n\\
&+(563{,}580+54{,}540\sqrt{85})\alpha^n+(563{,}580-54{,}540\sqrt{85})\beta^n-173{,}440(-1)^n\Big\},
\end{aligned}
$$

from which, after some algebra, (8.5.1) follows. □

It is clear that Ramanujan must have had a more insightful, more interesting, and less tedious proof. Since Ramanujan had previously devoted considerable effort to finding solutions to Euler's diophantine equation [49, 246, 249, 252], [65, pp. 224–226], [268], [39, pp. 197–200], [40, pp. 52, 54–56, 107–108], he could have used previously discovered parameterizations of solutions that he had found, probably along with recurrence relations, to establish Entry 8.5.1. One of Ramanujan's families of solutions [246, 268], [40, p. 56] to Euler's diophantine equation is given by

$$
(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3=(6a^2-4ab+4b^2)^3. \tag{8.5.5}
$$

Hirschhorn [141], [158]–[160] studied Ramanujan's claim over a period of several years and proposed that Ramanujan may have proceeded along the following lines. In [141], at the suggestion of Maurice Craig, the authors make the change of variables $a = A + B$ and $b = A - 2B$ to deduce (8.5.6) below. We now present a version of the proof that we just described.

Second Proof of Entry 8.5.1. Begin with the identity (8.5.5) and make the aforementioned changes of variable $a = A+B$ and $b = A-2B$ to deduce that

$$
(A^2+7AB-9B^2)^3+(2A^2-4AB+12B^2)^3=(2A^2+10B^2)^3+(A^2-9AB-B^2)^3. \tag{8.5.6}
$$

Define the sequence $\{h_n\}$ by

$$
h_{n+2} = 9h_{n+1} + h_n, \qquad n \geq 0, \qquad h_0 = 0, h_1 = 1. \tag{8.5.7}
$$

Then

$$
\begin{aligned}
h_{n+1}^2 - h_{n+2}h_n &= h_{n+1}^2 - (9h_{n+1} + h_n)h_n\\
&= -(h_n^2 - h_{n+1}h_{n-1})\\
&= \cdots = (-1)^n(h_1^2 - h_2h_0) = (-1)^n.
\end{aligned} \tag{8.5.8}
$$

Set

$$
A = h_{n+1} \qquad \text{and} \qquad B = h_n.
$$

Then, by (8.5.8),

$$
A^2 - 9AB - B^2 = h_{n+1}^2 - h_n(9h_{n+1} + h_n) = (-1)^n. \tag{8.5.9}
$$

Let

$$a_n = A^2 + 7AB - 9B^2 = h_{n+1}^2 + 7h_{n+1}h_n - 9h_n^2, \tag{8.5.10}$$
$$b_n = 2A^2 - 4AB + 12B^2 = 2h_{n+1}^2 - 4h_{n+1}h_n + 12h_n^2, \tag{8.5.11}$$
$$c_n = 2A^2 + 10B^2 = 2h_{n+1}^2 + 10h_n^2. \tag{8.5.12}$$

Then, by computer algebra and (8.5.9),

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Next we must show that a_n, b_n, and c_n satisfy the expansions given in Entry 8.5.1. The characteristic polynomial for the recurrence (8.5.7) is $x^2 - 9x - 1$ and its roots are $\frac{1}{2}(9 \pm \sqrt{85})$. Using the initial conditions given in (8.5.7), we can conclude from the general theory of linear recurrence relations that

$$h_n = \frac{1}{\sqrt{85}}\left\{\left(\frac{9+\sqrt{85}}{2}\right)^n - \left(\frac{9-\sqrt{85}}{2}\right)^n\right\},$$

and hence that

$$h_n^2 = \frac{1}{85}\left\{\left(\frac{83+9\sqrt{85}}{2}\right)^n + \left(\frac{83-9\sqrt{85}}{2}\right)^n - 2(-1)^n\right\},$$
$$h_{n+1}^2 = \frac{1}{85}\left\{\left(\frac{83+9\sqrt{85}}{2}\right)^{n+1} + \left(\frac{83-9\sqrt{85}}{2}\right)^{n+1} + 2(-1)^n\right\},$$

and

$$h_nh_{n+1} = \frac{1}{85}\left\{\left(\frac{9+\sqrt{85}}{2}\right)\left(\frac{83+9\sqrt{85}}{2}\right)^n + \left(\frac{9-\sqrt{85}}{2}\right)\left(\frac{83-9\sqrt{85}}{2}\right)^n - 9(-1)^n\right\}.$$

It follows that

$$\sum_{n=0}^{\infty} h_n^2 x^n = \frac{x-x^2}{1-82x-82x^2+x^3},$$
$$\sum_{n=0}^{\infty} h_{n+1}^2 x^n = \frac{1-x}{1-82x-82x^2+x^3},$$

and

$$\sum_{n=0}^{\infty} h_nh_{n+1} x^n = \frac{9x}{1-82x-82x^2+x^3}.$$

Hence, using (8.5.10)–(8.5.12), we easily find that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{(1-x)+7(9x)-9(x-x^2)}{1-82x-82x^2+x^3} = \frac{1+53x+9x^2}{1-82x-82x^2+x^3},$$
$$\sum_{n=0}^{\infty} b_n x^n = \frac{2(1-x)-4(9x)+12(x-x^2)}{1-82x-82x^2+x^3} = \frac{2-26x-12x^2}{1-82x-82x^2+x^3},$$

and

$$\sum_{n=0}^{\infty} c_n x^n = \frac{2(1-x)+10(x-x^2)}{1-82x-82x^2+x^3} = \frac{2+8x-10x^2}{1-82x-82x^2+x^3},$$

which are the claims of Ramanujan in Entry 8.5.1. □

Ramanujan also offers a companion to Entry 8.5.1.

Entry 8.5.2 (p. 341). *If*

$$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = \sum_{n=1}^{\infty} \alpha_{n-1} x^{-n},$$
$$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = \sum_{n=1}^{\infty} \beta_{n-1} x^{-n},$$

and

$$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = \sum_{n=1}^{\infty} \gamma_{n-1} x^{-n},$$

then

$$\alpha_n^3 + \beta_n^3 = \gamma_n^3 - (-1)^n. \tag{8.5.13}$$

We have corrected Ramanujan's formulation of (8.5.13), because he had written $+(-1)^n$ instead of $-(-1)^n$ in (8.5.13).

Proof. The proof follows along the same lines at that for Entry 8.5.1. Using the notation (8.5.2)–(8.5.4) and expanding the left-hand sides in Entry 8.5.2 into partial fractions, we are led to the formulas, for $n \geq 0$,

$$\alpha_n = A_1\beta^{n+1} + B_1\alpha^{n+1} + C_1(-1)^n,$$
$$\beta_n = A_2\beta^{n+1} + B_2\alpha^{n+1} + C_2(-1)^n,$$
$$\gamma_n = A_3\beta^{n+1} + B_3\alpha^{n+1} + C_3(-1)^n.$$

If we replace n by $n-1$, then, for $n \geq 1$, we are led to

$$\begin{aligned}
&\alpha_{n-1}^3+\beta_{n-1}^3+\gamma_{n-1}^3\\
&\quad=(A_1\beta^n+B_1\alpha^n-C_1(-1)^n)^3+(A_2\beta^n+B_2\alpha^n-C_2(-1)^n)^3\\
&\qquad-(A_3\beta^n+B_3\alpha^n-C_3(-1)^n)^3\\
&\quad=(A_1^3+A_2^3-A_3^3)\beta^{3n}+(B_1^3+B_2^3-B_3^3)\alpha^{3n}\\
&\qquad+(-C_1^3-C_2^3+C_3^3-6A_1B_1C_1-6A_2B_2C_2+6A_3B_3C_3)(-1)^n\\
&\qquad+(-3A_1^2C_1-3A_2^2C_2+3A_3^2C_3)(-1)^n\beta^{2n}\\
&\qquad+(-3B_1^2C_1-3B_2^2C_2+3B_3^3C_3)(-1)^n\alpha^{2n}\\
&\qquad+(3A_1^2B_1+3A_2^2B_2-3A_3^2B_3+3A_1C_1^2+3A_2C_2^2-3A_3C_3^2)\beta^n\\
&\qquad+(3A_1B_1^2+3A_2B_2^2-3A_3B_3^2+3B_1C_1^2+3B_2C_2^2-3B_3C_3^2)\alpha^n\\
&\quad=(-1)^n,
\end{aligned}$$

which is what we wanted to prove. □

In [159], Hirschhorn used an idea of D. Zeilberger to show that it suffices to check the first seven cases of (8.5.1).

A beautiful generalization of Entry 8.5.1 has been developed by J. McLaughlin [222], who established the following theorem.

Theorem 8.5.1. *Define* 11 *sequences of integers* a_k, b_k, c_k, d_k, e_k, f_k, p_k, q_k, r_k, s_k, *and* t_k, $k \geq 0$, *by*

$$\frac{x^2+164x+3}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}a_kx^k,\qquad \frac{-5x^2+138x+3}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}p_kx^k,$$

$$\frac{-7x^2+134x+1}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}b_kx^k,\qquad \frac{3x^2+244x+1}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}q_kx^k,$$

$$\frac{-x^2+298x-1}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}c_kx^k,\qquad \frac{x^2+254x-7}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}r_kx^k,$$

$$\frac{-5x^2+228x-7}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}d_kx^k,\qquad \frac{-7x^2+148x-5}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}s_kx^k,$$

$$\frac{3x^2+258x-5}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}e_kx^k,\qquad \frac{3}{1-x}=:\sum_{k=0}^{\infty}t_kx^k,$$

$$\frac{-3x^2+94x-3}{x^3-99x^2+99x-1}=:\sum_{k=0}^{\infty}f_kx^k.$$

Then, for $1 \leq j \leq 5$ *and each* $k \geq 0$,

$$a_k^j+b_k^j+c_k^j+d_k^j+e_k^j+f_k^j-p_k^j-q_k^j-r_k^j-s_k^j-t_k^j=1. \tag{8.5.14}$$

Note that (8.5.14) holds for *each* integer j, $1 \leq j \leq 5$, in contrast to Ramanujan's theorem, in which the exponent is fixed at 3. McLaughlin provides the following example to illustrate his theorem. If

$$\{a_1, b_1, c_1, d_1, e_1, f_1, p_1, q_1, r_1, s_1, t_1\} = \{-461, -233, -199, 465, 237, 203, -435, -343, 439, 347, 3\},$$

then

$$(-461)^j + (-233)^j + (-199)^j + 465^j + 237^j + 203^j - (-435)^j - (-343)^j - 439^j - 347^j - 3^j = 1,$$

for $1 \leq j \leq 5$.

Following Entry 8.5.1, Ramanujan records the following six examples:

Entry 8.5.3 (p. 341).

$$\begin{aligned} 9^3 + 10^3 &= 12^3 + 1, \\ 6^3 + 8^3 &= 9^3 - 1, \\ 135^3 + 138^3 &= 172^3 - 1, \\ 11{,}161^3 + 11{,}468^3 &= 14{,}258^3 + 1, \\ 791^3 + 812^3 &= 1{,}010^3 - 1, \\ 65{,}601^3 + 67{,}402^3 &= 83{,}802^3 + 1. \end{aligned}$$

Readers will immediately recognize the taxicab-number representations in the first example above. For further information about the taxicab-number 1729 in Ramanujan's work, consult the second author's book [39, pp. 199–200].

8.6 On the Divisors of $N!$

On page 326 of [269], Ramanujan offers four disparate claims in the theory of numbers. We examine three of them in this and the following two sections. (The remaining claim is discussed in Chap. 6 of [14].)

Except for notation, we quote Ramanujan in his first claim.

Entry 8.6.1 (p. 326). *If $d(N!)$ be the no. of divisors of $N!$, then*

$$C^{\frac{N}{\log N}(1-\epsilon)} < d(N!) < C^{\frac{N}{\log N}(1+\epsilon)} \tag{8.6.1}$$

where

$$C = (1+1)\sqrt{1+\tfrac{1}{2}}\sqrt[3]{1+\tfrac{1}{3}}\sqrt[4]{1+\tfrac{1}{4}}\sqrt[5]{1+\tfrac{1}{5}}\cdots. \tag{8.6.2}$$

To the right of (8.6.1) in [269] is an appended note that reads, "old arithmetical conjecture," which was probably written by Hardy. We are unable to trace the origin of this conjecture. However, in fact, Ramanujan proved a more precise version of (8.6.1) in his paper [259, Eqs. (265)–(267)], [267, p. 127]. Namely, he proved that [259, Eq. (266)]

$$d(N!) = C^{\frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right)}.$$

P. Erdős, S.W. Graham, A. Ivić, and C. Pomerance [116, p. 339] have, in fact, established an asymptotic series for $d(N!)$.

Theorem 8.6.1. *Define the sequence of integrals c_k, $k \geq 0$, by*

$$c_k = \int_1^\infty \frac{\log([t]+1)}{t^2} \log^k t\,dt. \tag{8.6.3}$$

Then, for any fixed integer $K \geq 0$,

$$d(n!) = \exp\left\{ \frac{n}{\log n} \sum_{k=0}^{K} \frac{c_k}{\log^k n} + O\left(\frac{n}{\log^{K+2} n}\right) \right\}. \tag{8.6.4}$$

In particular,

$$c_0 = \sum_{k=2}^\infty \frac{\log k}{k(k-1)} \approx 1.25775. \tag{8.6.5}$$

In order for Ramanujan's claim (8.6.1) to be compatible with (8.6.4) and (8.6.3), in particular with (8.6.5), it would be required that

$$\log C = c_0. \tag{8.6.6}$$

To that end, as observed first by M. Tip Phaovibul, we see that

$$\begin{aligned} \log C = \sum_{k=1}^\infty \frac{1}{k} \log\left(1 + \frac{1}{k}\right) &= \sum_{k=1}^\infty \frac{1}{k} \{\log(k+1) - \log k\} \\ &= \sum_{k=2}^\infty \left(\frac{1}{k-1} - \frac{1}{k}\right) \log k = \sum_{k=2}^\infty \frac{\log k}{k(k-1)} = c_0. \end{aligned}$$

8.7 Sums of Two Squares

We quote from the second claim on page 326 in [269].

Entry 8.7.1 (p. 326). *If $S(N)$ be the no. of integers in which N can be expressed as the sum of 2 squares, then the maximum order of $S(N)$*

$$= \sqrt{\text{max. order of } d(N^2 + aN + b)} \cdot e^{O(\log N)^{1/2+\epsilon}}. \tag{8.7.1}$$

Perhaps needless to say, "the number of integers" is best replaced by "the number of ways." As Ramanujan must have indeed also assumed, we take a and b to be positive integers. The error term in (8.7.1) is under the assumption of the Riemann Hypothesis. Our attention therefore is focused on "max order" on both sides of (8.7.1).

In the sequel, we frequently employ the following inequalities for the logarithmic integral $\mathrm{Li}(x)$ without comment:

$$\begin{aligned}\mathrm{Li}(2x) &= \frac{2x}{\log x} + \frac{2(1-\log 2)x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right) \\ &\leq 2\mathrm{Li}(x) = \frac{2x}{\log x} + \frac{2x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right). \end{aligned} \tag{8.7.2}$$

We first address the maximum order of $S(N)$, for which the notation $Q_2(N)$ is used in Ramanujan's unpublished manuscript that was to form part of his paper [259]. This handwritten manuscript was published with the lost notebook [269, pp 281–308] in 1988, while an annotated version was published in the first volume of the *Ramanujan Journal* by J.-L. Nicolas and G. Robin [233]. It was revised and included as Chap. 10 in the present authors' third volume on the lost notebook [14]. Using the prime number theorem, Ramanujan showed that the maximum order of $Q_2(N)$ is

$$\begin{aligned}\text{max. order of } Q_2(N) &= 2^{\frac{1}{2}\mathrm{Li}(2\log N)+O\left\{\log N e^{-a\sqrt{\log N}}\right\}} \\ &= 2^{(1+o(1))\frac{\log N}{\log\log N}}, \end{aligned} \tag{8.7.3}$$

where a is a positive constant. The Q_2-highly composite numbers were defined by Ramanujan in [233], [14, Chap. 10]. A number N is Q_2-highly composite if whenever $M < N$, then $Q_2(M) < Q_2(N)$. They are of the form [14, Sect. 52, p. 362]

$$N = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}, \tag{8.7.4}$$

where $q_1 = 5, q_2 = 13, \ldots$ are the primes congruent to 1 modulo 4 in increasing order. For such numbers N, $Q_2(N) = d(N)$.

Now suppose that $a = b = 0$ in (8.7.1). Then Ramanujan proved in [259, Sect. 49, Eq. (256)] that

$$\text{max. order of } d(N^2) = 3^{\mathrm{Li}\left(\frac{1}{2}\log N^2\right)} = 3^{(1+o(1))\frac{\log N}{\log\log N}}. \tag{8.7.5}$$

Hence, in comparing (8.7.3) with (8.7.5), we see that

$$\text{max. order of } Q_2(N) > \sqrt{\text{max. order of } d(N^2+aN+b)} \cdot e^{O(\log N)^{1/2+\epsilon}}, \tag{8.7.6}$$

for $a = b = 0$. Thus, in this instance, Ramanujan's assertion in Entry 8.7.1 is incorrect.

In general, Ramanujan proved that for infinitely many n [259, Sect. 5], [267, p. 86],

$$d(n) = 2^{(1+o(1))\frac{\log N}{\log\log N}}. \tag{8.7.7}$$

Thus, if N^2+aN+b factors over $\mathbb{Q}$ into $(N+r_1)(N+r_2)$ and we apply (8.7.7) to each of the factors, then [259, Sect. 49, Eq. (256)]

$$d(N^2+aN+b) \leq d(N+r_1)d(N+r_2) = 2^{2(1+o(1))\frac{\log N}{\log\log N}}, \tag{8.7.8}$$

and so we see, in view of (8.7.3), that Ramanujan's claim in Entry 8.7.1 *may* hold. Of course, (8.7.5) shows that, at least in this special case, there is a strict inequality in (8.7.8), vitiating the possible validity of Entry 8.7.1.

Let us now discuss the case that N^2+aN+b is irreducible, in which case we might expect that $d(N^2+aN+b)$ is "small." Consider the case $a=0$ and $b=1$. The prime factors of N^2+1 are all congruent to 1 modulo 4, and so, invoking (8.7.3) twice, we find that

$$\begin{aligned} d(N^2+1) = Q_2(N^2+1) &\leq \text{max. order of } Q_2(N^2+1) \\ &\approx 2^{\frac{1}{2}\mathrm{Li}(2(\log(N^2+1)))} \approx 2^{\frac{1}{2}\mathrm{Li}(4\log N)} \\ &\leq 2^{\mathrm{Li}(2\log N)} = 2^{2(1+o(1))\frac{\log N}{\log\log N}}. \end{aligned} \tag{8.7.9}$$

Thus, again, we see the possibility of Entry 8.7.1 being valid. However, the first inequality of (8.7.9) is likely quite crude, for it is possible that N^2+1 has relatively few divisors.

In summary, we see that Entry 8.7.1 is likely valid for some choices of a and b but not valid for other choices. In contrast to Ramanujan's assertion, it seems probable that in all cases,

$$\text{max. order of } Q_2(N) \geq \sqrt{\text{max. order of } d(N^2+aN+b)} \cdot e^{O(\log N)^{1/2+\epsilon}}. \tag{8.7.10}$$

8.8 A Lattice Point Problem

On page 326 of [269], Ramanujan considers an analogue of the famous *circle problem* for higher powers. For each positive integer $k \geq 2$, let

$$R_{k,2}(x) := \sum_{\substack{m^k+n^k\leq x \\ m,n\geq 0}} 1, \tag{8.8.1}$$

where Ramanujan does not consider m^k+n^k and n^k+m^k to be distinct. He then asserts the following asymptotic formula.

Entry 8.8.1 (p. 326).

$$R_{k,2}(x) = \frac{x^{2/k}}{4k}\frac{\{\Gamma(1/k)\}^2}{\Gamma(2/k)} + O\left(x^{1/k+\epsilon}\right), \tag{8.8.2}$$

as $x \to \infty$, for each fixed $\epsilon > 0$.

In a parenthetical remark after (8.8.2), Ramanujan writes, "Assuming the equation $x^n + y^n = u^n + v^n$ (x and y being different from u and v) has not got an infinite no. of solutions." Perhaps Ramanujan wanted to determine the number of *distinct* integers N that are the sum of 2 kth powers. If fact, if there were only finitely many solutions to $u^k + v^k = x^k + y^k$, then this number would equal $R_{k,2}(x) + O(1)$.

We are uncertain who first considered this lattice point problem. The earliest reference known to us is a paper by J.G. van der Corput [100] in 1923. Van der Corput [100], E. Krätzel, in a series of three papers [197–199], and B. Randol [270] proved that for $k \geq 3$,

$$R_{k,2}(x) = \frac{x^{2/k}}{4k} \frac{\{\Gamma(1/k)\}^2}{\Gamma(2/k)} + O\left(x^{1/k-1/k^2}\right). \tag{8.8.3}$$

Moreover, in [199], Krätzel showed that the error term in (8.8.3) is sharp. More precisely, he proved that

$$R_{k,2}(x) = \frac{x^{2/k}}{4k} \frac{\{\Gamma(1/k)\}^2}{\Gamma(2/k)} + \Omega\left(x^{1/k-1/k^2}\right).$$

8.9 Mersenne Numbers

A *Mersenne number* is a number of the form $2^p - 1$, where p is a prime. If in addition $2^p - 1$ is prime, then it is called a *Mersenne prime*. In a two-page manuscript published with the lost notebook [269, pp. 259–260], Ramanujan uses a different definition for a Mersenne prime; he calls p, not $2^p - 1$, a Mersenne prime when the latter is prime. We adhere here to Ramanujan's definition. So, the first Mersenne primes are $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1{,}279, 2{,}203, 2{,}281, 3{,}217$. As of this writing, 47 Mersenne primes are known. A famous and long outstanding conjecture is that there are infinitely many Mersenne primes.

The aforementioned pages 259–260 are not part of the original lost notebook, and so it is difficult to assess precisely when they were written. None of Ramanujan's claims on these two pages are correct, as observed by P.G. Brown [81], who evidently was the first person to examine the pages and write about them in print. However, most likely, Ramanujan knew that his thoughts were speculative and so wanted to stimulate further discussion and computation. At the time of his writing, very few Mersenne primes had been computed. In fact, the 10th, 11th, and 12th were computed in 1911, 1914, and 1876, respectively, and so he had at most a list of 12 Mersenne primes at his disposal. With such little numerical data, Ramanujan certainly was aware that any assertions he might make would be tenuous at best. Thus, ending the paper [81] with the statement, "Clearly even the great Ramanujan had his 'bad days'" seems uncharitable, since Ramanujan was undoubtedly trying to stimulate discussion.

Ramanujan thought that Mersenne primes were either of the form $a^2 + ab + b^2$ or $a^2 + b^2$. It is well known that every odd prime p that is congruent to 1 modulo 4 can be represented by the form $a^2 + b^2$ [234, p. 164] and that every prime p with $p \equiv 1 \pmod 6$ can be represented by the form $a^2 + ab + b^2$ [234, p. 176]. Thus, Ramanujan speculated that there are no Mersenne primes congruent to 11 modulo 12. Among the 12 Mersenne primes that were known up to Ramanujan's death in 1920, only 107 is congruent to 11 modulo 12, and since 107 was not discovered until 1914 by R.E. Powers, it is not clear that at the time Ramanujan recorded his speculation, he even knew of this Mersenne prime. In fact, the second Mersenne prime congruent to 11 modulo 12 is the 28th Mersennne prime, namely 86,243, which was discovered by D. Slowinski in 1982. Of the 45 known Mersenne primes exceeding 3, only 5 are $\equiv 11 \pmod{12}$. On the other hand, 9 are congruent to 1 modulo 12, 18 are congruent to 5 modulo 12, and 13 are congruent to 7 modulo 12. On the basis of this limited data, it could be speculated that the Mersenne primes congruent to 11 modulo 12 have smaller density than those in the remaining three residue classes modulo 12. Other than the fact that the quadratic forms $a^2 + ab + b^2$ and $a^2 + b^2$ avoid primes congruent to 11 modulo 12, we do not have any explanation for Ramanujan's bringing these forms into the theory of Mersenne primes. Nonetheless, representations of Mersenne primes by the quadratic form $x^2 + 7y^2$ are relevant. In particular, if $M_\ell = 2^{\ell-1}$ is a Mersenne prime with $\ell \equiv 1 \pmod 3$ and we write $M_\ell = x^2 + 7y^2$, where x and y are integers, then x is divisible by 8 [213].

We now quote Ramanujan's ten statements on Mersenne primes and comment on each. Most of our remarks are those supplied by Brown [81].

1. "All Mersenne's primes are either of the form $a^2 + ab + b^2$ or of the form $a^2 + b^2$. Then since a number of the form $12n - 1$ cannot be expressed in any one of the above two forms, we infer that"

   As we indicated above, 107 provides a counterexample, since $2^{107} - 1$ is prime, and as we also indicated above, Ramanujan may likely not have had access to this fact when he wrote.

2. "A Mersenne's prime is never of the form $12k - 1$. Thus for example $2^{11}-1$, $2^{23}-1$, $2^{47}-1$, $2^{59}-1$, $2^{71}-1$, $2^{83}-1$, $2^{107}-1$, $2^{131}-1$, $2^{167}-1$, $2^{179}-1$, $2^{191}-1$, $2^{227}-1$, $2^{239}-1$, $2^{251}-1$, etc. should be composite numbers. Hence we may divide all Mersenne primes into two classes, one comprising primes that can be expressed as $a^2 + ab + b^2$ and the other containing primes that cannot be expressed as $a^2 + ab + b^2$."

   It is unclear how many of these numbers Ramanujan had actually calculated.

3. "Hence the Mersenne's primes of the 1st class except 1 and 3 are of the form $6n+1$, while those of the 2nd except 2 are of the form $12n+5$. Thus we have,

Nos. of the 1st class : $-1, 3, 7, 13, 19, 31, 61, 127$, etc.

Nos. of the 2nd class : $-2, 5, 17, 89, 257$, etc."

Observe that $2^{257} - 1$ is not prime. It is curious that Mersenne made the same mistake.

4. "Theorem. If P be any prime, and p any odd prime and if either of $\frac{P^p-1}{P-1}$ or $\frac{P^p-1}{p(P-1)}$ happens to be a prime, then that prime will be a Mersenne's prime of the 1st class. As a particular case we have when $p = 3$."

   This claim is incorrect. For example,

$$\frac{7^5 - 1}{7 - 1} = 2{,}801,$$

   which is prime. However, $2^{2,801} - 1$ is composite. Also,

$$\frac{31^3 - 1}{3(31 - 1)} = 331,$$

   which is prime. However, $2^{331} - 1$ is composite. It might be remarked that one can find the factorizations of $2^n - 1$ for $n < 1{,}200$ in [78].

5. "If P be any prime and if either of $P^2 + P + 1$ or $\frac{P^2+P+1}{3}$ happens to be a prime, then that prime will be a Mersenne's prime of the 1st class. As another particular case when $P = 2$ we have"

   This claim is also not true. For example, $17^2 + 17 + 1 = 307$, which is prime, but $2^{307} - 1$ is composite. Also, $\frac{31^2+31+1}{3} = 331$, which, as seen above, is not a Mersenne prime.

6. "If p be a Mersenne's prime then $2^p - 1$ will be a Mersenne's prime of the 1st class. As examples of (5) and (6) we have
   $1^2 + 1 + 1 = 3$; $2^2 + 2 + 1 = 7$; $3^2 + 3 + 1 = 13$; $5^2 + 5 + 1 = 31$; $\frac{7^2+7+1}{3} = 19$; ($11^2 + 11 + 1 = 133$ composite); $\frac{13^2+13+1}{3} = 61$; $17^2 + 17 + 1 = 307$; $\frac{19^2+19+1}{3} = 127$; and so on. Again $2^2 - 1 = 3$; hence $2^3 - 1 = 7$ a prime; hence $2^7 - 1 = 127$ a prime; hence $2^{127} - 1$ is a prime. $2^5 - 1 = 31$; hence $2^{31} - 1$ is prime."

   This statement is false. For example, 13 is a Mersenne prime, but $2^{13} - 1$ is not.

7. "From (3) we can infer that the number of Mersenne's primes of the 2nd class is always about $\frac{1}{2}$ of the number of those of the 1st class. There may be a general theorem like (4) for the Mersenne's primes of the 2nd class of which the particular case analogous to (6) will be."

   Of the first 44 Mersenne primes exceeding 3, 22 are in the 1st class and 17 are in the 2nd class, which does not support Ramanujan's speculation. As indicated earlier, 5 is not in either class.

8. "If $2^p + 1$ be a prime, then $2^p + 1$ will be a Mersenne's prime of the 2nd class. Thus for example we have

   2+1=3; hence $2^3 - 1$ is a prime; $2^2 + 1 = 5$ hence $2^5 - 1$ is a prime; $2^4 + 1 = 17$ hence $2^{17} - 1$ is a prime; $2^8 + 1 = 257$; hence $2^{257} - 1$ is a prime and so on."

   This claim is false, because, as we remarked earlier, $2^{257} - 1$ is not prime.

9. "Mersenne's primes of the 2nd class are always of the form $(2^a)^2 + (4b+1)^2$ where a assumes all integral values, 0, 1, 2, 3 etc. without an exception, b is a positive integer including 0, $4b + 1$ is never greater than 2^a and for every value of a, there is at least one value of b. Thus we have, when $a = 0$, $b = 0$ hence $2^2 - 1$ is a prime; when $a = 1$, b must be 0 hence $2^5 - 1$ is a prime; when $a = 2$, b must be 0 hence $2^{17} - 1$ is a prime; when $a = 3$, b may be 0 or 1, but when $b = 0$, $(2^8)^2 + 1$ becomes composite, hence b must be 1 since $(2^3)^2 + 5^2$ is a prime, hence $2^{89} - 1$ is a prime."

   This statement is also false. Consider the prime $4{,}253 \equiv 5 \pmod{12}$. Then $2^{4{,}253} - 1$ is prime, and we have the unique representation (up to order and the signs of the summands) $4{,}253 = 53^2 + 38^2$, but neither summand is a power of 2. Furthermore, if $a = 4$, then $b = 0, 1, 2$, or 3, and then $(2^a)^2 + (4b + 1)^2 = 257, 281, 337, 425$, respectively, and none of these four numbers is a Mersenne prime. It should be noted that the Mersenne prime 4,253 was not discovered until 1961.

10. "Another theorem analogous to (8) is, if $2^p + 1$ is a prime then $2^{2^p} + 1$ is also a prime."

    A counterexample to this claim is given by $2^8 + 1$, which is prime, but $2^{2^8} + 1$ is composite.

9

Divisor Sums

9.1 Introduction

Pages 270 and 271 in [269] are devoted to sums involving $\sigma_k(n)$, the sum of the kth powers of the divisors of the positive integer n. At the top of the page is a note, possibly written by Gertrude Stanley, indicating that these pages were intended to be a conclusion of Ramanujan's paper [265], [267, pp. 179–199]. Indeed, this is most certainly true. In the upper right-hand corners of the two pages are the numbers (29) and (30), respectively, and the pages are written in ¶18; the last section of [265] is numbered 17. We do not know why Ramanujan deleted Sect. 18 from his paper, but perhaps he thought that the content of Sect. 18 strayed slightly from that of the remainder of the paper. The last result in the omitted section appears to be incorrect, but it is easily corrected. The two primary results in this section are of the same type as five theorems on pages 277 and 278 of his second notebook [268], which were first proved in print by P. Bialek and the second author [45]. These proofs can also be found in [41, pp. 426–444]. Although, as usual, he did not supply hypotheses for the aforementioned theorems in his notebooks, in the partial manuscript at hand, Ramanujan indicates that his proofs are valid for (real) $s > 2$, while in [268] and [45], $s = n$ is restricted to be an integer.

Pages 272 and 273 of [269] also have some relation to [265], and so evidently for this reason, the editors placed the pages at this juncture. We discuss these pages in Sect. 9.4.

Page 255 is also connected with ¶17 of [265]. However, Ramanujan evidently did not include this material in his paper because his claims are imprecisely stated. We have put two of them on a firmer foundation.

In Sect. 9.6, we examine a short, elementary partial manuscript on the divisor function $d(n)$. The results proved here are at the level of exercises in an introductory graduate (or undergraduate) course on analytic number theory.

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Lastly, page 368 is an isolated page that we address in Sect. 9.7. It is devoted to some of Ramanujan's musings on the Dirichlet divisor problem, which we discuss in detail in Chap. 2.

9.2 Ramanujan's Conclusion to [265]

We first record Ramanujan's Sect. 18 on pages 270 and 271 of [269] exactly as he wrote it. Then in Sect. 9.3, we supply more details, as well as further comments.

I shall conclude this paper by finding an expression for

$$\sum_{n=1}^{\infty} \sigma_s(n)x^n, \qquad \sum_{n=1}^{\infty} r_s(n)x^n$$

which shows the asymptotic nature for large values of s. If $\mathrm{Re}(x) > 0$, $\mathrm{Re}(x)$ being the real part of x, it is well known that

$$\pi + 2\pi(e^{-2\pi x} + e^{-4\pi x} + e^{-6\pi x} + \cdots) \\ = \frac{1}{x} + \frac{1}{x+i} + \frac{1}{x-i} + \frac{1}{x+2i} + \frac{1}{x-2i} + \cdots . \tag{9.2.1}$$

Differentiating the two sides of (9.2.1) $s-1$ times we obtain

$$1^{s-1}e^{-2\pi x} + 2^{s-1}e^{-4\pi x} + 3^{s-1}e^{-6\pi x} + \cdots \\ = \frac{\Gamma(s)}{(2\pi)^s}\left\{\frac{1}{x^s} + \frac{1}{(x+i)^s} + \frac{1}{(x-i)^s} + \frac{1}{(x+2i)^s} + \cdots\right\} \tag{9.2.2}$$

if s is an integer greater than 1. This result is quite elementary. But it can be shown by the theory of residues that (9.2.2) is true for *all* values of s greater than 1. It follows from this that

$$\sum_{n=1}^{\infty} \sigma_{s-1}(n)e^{-2\pi nx} = \sum_{n=1}^{\infty}(1^{s-1}e^{-2\pi nx} + 2^{s-1}e^{-4\pi nx} + \cdots) \\ = \frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\left\{\frac{1}{(nx)^s} + \frac{1}{(nx+i)^s} + \frac{1}{(nx-i)^s} + \cdots\right\}.$$

The double series in the last equation is absolutely convergent if $s > 2$. From this we easily deduce that

$$\sigma_{s-1}(1)e^{-2\pi x} + \sigma_{s-1}(2)e^{-4\pi x} + \cdots \\ = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{\frac{1}{x^s} + \sum\frac{1}{(\mu x+\nu i)^s} + \sum\frac{1}{(\mu x-\nu i)^s}\right\} \\ = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{\frac{1}{x^s} + 2\sum\frac{\cos\left(s\tan^{-1}\frac{\nu}{\mu x}\right)}{(\mu^2x^2+\nu^2)^{\frac{1}{2}s}}\right\}, \tag{9.2.3}$$

where $s > 2$ and μ and ν assume all positive integral values which are prime to each other.

Let us consider some particular cases of (9.2.3). Suppose $x = 1$. Then

$$2\sum \frac{\cos\left(s\tan^{-1}\frac{\nu}{\mu}\right)}{(\mu^2+\nu^2)^{\frac{1}{2}s}} = \sum \frac{\cos\left(s\tan^{-1}\frac{\nu}{\mu}\right)+\cos\left(s\tan^{-1}\frac{\mu}{\nu}\right)}{(\mu^2+\nu^2)^{\frac{1}{2}s}}$$
$$= 2\cos\frac{\pi s}{4}\sum \frac{\cos\left(s\tan^{-1}\frac{\mu-\nu}{\mu+\nu}\right)}{(\mu^2+\nu^2)^{\frac{1}{2}s}}.$$

It follows that if $s > 2$, then

$$\begin{aligned}
&\sigma_{s-1}(1)e^{-2\pi}+\sigma_{s-1}(2)e^{-4\pi}+\sigma_{s-1}(3)e^{-6\pi}+\cdots \qquad (9.2.4)\\
&=\frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{1+2\cos\frac{\pi s}{4}\sum\frac{\cos\left(s\tan^{-1}\frac{\mu-\nu}{\mu+\nu}\right)}{(\mu^2+\nu^2)^{\frac{1}{2}s}}\right\}\\
&=\frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{1+2\cos\frac{\pi s}{4}\left(\frac{1}{2^{\frac{1}{2}s}}+\frac{2\cos\left(s\tan^{-1}\frac{1}{3}\right)}{5^{\frac{1}{2}s}}\right.\right.\\
&\qquad\left.\left.+\frac{2\cos\left(s\tan^{-1}\frac{1}{2}\right)}{10^{\frac{1}{2}s}}+\frac{2\cos\left(s\tan^{-1}\frac{1}{5}\right)}{13^{\frac{1}{2}s}}+\cdots\right)\right\},
\end{aligned}$$

where μ and ν are positive integers that are prime to each other and 2, 5, 10, $13, \ldots$ are sums of two squares that are prime to each other.

Similarly putting $x = \frac{1}{2}(\sqrt{3}+i)$ in (9.2.3) we can show that if $s > 2$ then

$$\begin{aligned}
&\sigma_{s-1}(1)e^{-\pi\sqrt{3}}-\sigma_{s-1}(2)e^{-2\pi\sqrt{3}}+\sigma_{s-1}(3)e^{-3\pi\sqrt{3}}-\cdots=-2\frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\\
&\times\left\{\cos\frac{\pi s}{6}+2\cos\frac{\pi(s+1)}{6}\cos\frac{\pi(s-1)}{6}\sum\frac{\cos\left(s\tan^{-1}\frac{K}{\lambda\sqrt{3}}\right)}{(\mu^2+\mu\nu+\nu^2)^{\frac{1}{2}s}}\right\}. \qquad (9.2.5)
\end{aligned}$$

9.3 Proofs and Commentary

Up to the two examples given by Ramanujan, the argument is straightforward. The identity (9.2.1) is equivalent to the partial fraction decomposition of $\coth(\pi x)$. Ramanujan's remark that (9.2.2) is valid for $s > 1$ is correct. In fact, this more general version is called the Lipschitz summation formula [183, p. 65].

Let us consider the first example in which $x = 1$. To obtain the second equality in the display prior to (9.2.4), we use elementary trigonometry. To that end,

$$\begin{aligned}
&\cos\left(s\tan^{-1}\frac{\nu}{\mu}\right)+\cos\left(s\tan^{-1}\frac{\mu}{\nu}\right)\\
&\quad=\cos\left(s\tan^{-1}\frac{\nu}{\mu}\right)+\cos\left(s\left\{\frac{\pi}{2}-\tan^{-1}\frac{\nu}{\mu}\right\}\right)\\
&\quad=2\cos\left(\frac{\pi s}{4}\right)\cos\left(s\left\{\frac{\pi}{4}-\tan^{-1}\frac{\nu}{\mu}\right\}\right).
\end{aligned}\tag{9.3.1}$$

However,

$$\frac{\pi}{4}-\tan^{-1}\frac{\nu}{\mu}=\tan^{-1}1-\tan^{-1}\frac{\nu}{\mu}=\tan^{-1}\left(\frac{1-\nu/\mu}{1+\nu/\mu}\right)=\tan^{-1}\frac{\mu-\nu}{\mu+\nu}.\tag{9.3.2}$$

If we put (9.3.2) into (9.3.1), we find that the proof of (9.2.4) is finished.

Note that on the right-hand side of (9.2.5) there are two undefined constants, K and λ. Unfortunately, we are unable either to identify them or to obtain an identity of the form given by Ramanujan. However, we are able to obtain an identity for the left side of (9.2.5), which appears to be somewhat simpler than that given by Ramanujan.

Using the first equality in (9.2.3), following Ramanujan, we set $x=e^{\pi i/6}$, but we also invoke (9.2.3) a second time, now with $x=e^{-\pi i/6}$. For the first application, we need the elementary calculations

$$\begin{aligned}
(\mu e^{\pi i/6}+\nu i)^s&=(\mu^2+\mu\nu+\nu^2)^{\frac12 s}\exp\left(is\tan^{-1}\frac{\mu+2\nu}{\sqrt{3}\,\mu}\right),\\
(\mu e^{\pi i/6}-\nu i)^s&=(\mu^2-\mu\nu+\nu^2)^{\frac12 s}\exp\left(is\tan^{-1}\frac{\mu-2\nu}{\sqrt{3}\,\mu}\right),
\end{aligned}$$

and for the second application, we need

$$\begin{aligned}
(\mu e^{-\pi i/6}+\nu i)^s&=(\mu^2-\mu\nu+\nu^2)^{\frac12 s}\exp\left(is\tan^{-1}\frac{-\mu+2\nu}{\sqrt{3}\,\mu}\right),\\
(\mu e^{-\pi i/6}-\nu i)^s&=(\mu^2+\mu\nu+\nu^2)^{\frac12 s}\exp\left(-is\tan^{-1}\frac{\mu+2\nu}{\sqrt{3}\,\mu}\right).
\end{aligned}$$

Using these calculations, we now add the two equalities arising from (9.2.3) and find that

$$\begin{aligned}
2\sum_{n=1}^{\infty}(-1)^n\sigma_{s-1}(n)e^{-n\pi\sqrt{3}}&=\frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\Bigg\{2\cos\left(\frac{\pi s}{6}\right)\\
&+\sum_{\substack{\mu,\nu=1\\(\mu,\nu)=1}}^{\infty}\frac{2\cos\left(s\tan^{-1}\dfrac{\mu+2\nu}{\sqrt{3}\,\mu}\right)}{(\mu^2+\mu\nu+\nu^2)^{\frac12 s}}+\sum_{\substack{\mu,\nu=1\\(\mu,\nu)=1}}^{\infty}\frac{2\cos\left(s\tan^{-1}\dfrac{\mu-2\nu}{\sqrt{3}\,\mu}\right)}{(\mu^2-\mu\nu+\nu^2)^{\frac12 s}}\Bigg\}.
\end{aligned}\tag{9.3.3}$$

We now replace ν by $-\nu$ in the second sum on the right-hand side of (9.3.3), observe that the first expression on the right-hand side of (9.3.3) corresponds to the term $\nu = 0$ in either double sum, and lastly divide both sides by 2. Hence, we conclude that

$$\sum_{n=1}^{\infty}(-1)^n\sigma_{s-1}(n)e^{-n\pi\sqrt{3}} = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\sum_{\substack{\mu=1,\nu=-\infty\\(\mu,\nu)=1}}^{\infty}\frac{\cos\left(s\tan^{-1}\dfrac{\mu+2\nu}{\sqrt{3}\,\mu}\right)}{(\mu^2+\mu\nu+\nu^2)^{\frac{1}{2}s}}.$$

9.4 Two Further Pages on Divisors and Sums of Squares

Some of the pages published with the lost notebook have been numbered and arranged in consecutive runs, especially in the latter portions of [269]. We do not know who provided the numbering, but it was not Ramanujan and most likely not G.H. Hardy either. After pages 270 and 271, there is a string of nine consecutive pages beginning with number 22. The editors evidently designated pages 22 and 23 as pages 272 and 273 in [269], because their content pertains to pages 270 and 271. We first discuss page 273, which contains two asymptotic formulas. If $k \mid n$, then in the sequel we interpret

$$\frac{\sin(\pi n)}{\sin(\frac{1}{k}\pi n)} = \lim_{x\to n}\frac{\sin(\pi x)}{\sin(\frac{1}{k}\pi x)}.$$

Entry 9.4.1 (p. 273). *If $s > 1$, then, as $n \to \infty$,*

$$\sigma_s(n) = n^s\sin(\pi n)\left\{\frac{1}{1^{s+1}\sin(\pi n)} + \frac{1}{2^{s+1}\tan(\frac{1}{2}\pi n)} + \frac{1}{3^{s+1}\sin(\frac{1}{3}\pi n)} + \frac{1}{4^{s+1}\tan(\frac{1}{4}\pi n)} + \frac{1}{5^{s+1}\sin(\frac{1}{5}\pi n)} + \frac{1}{6^{s+1}\tan(\frac{1}{6}\pi n)} + \cdots\right\}.$$

Entry 9.4.1 is identical to (14.3) in Ramanujan's paper [265], [267, p. 193].

Entry 9.4.2 (p. 273). *Let s be an integer greater than 1, and let $r_{2s}(n)$ denote the number of representations of the positive integer n as a sum of $2s$ squares. Furthermore, set*

$$R(s) := \sum_{k=0}^{\infty}\frac{\cos(k\pi s)}{(2k+1)^s}, \qquad s > 1.$$

Then, as $n \to \infty$,

$$r_{2s}(n) = \frac{\pi^s n^{s-1}}{(s-1)!R(s)}\sin(\pi n)\left\{\frac{1}{1^s\sin(\pi n)} + \frac{1}{2^s\sin(\frac{1}{2}\pi n + \frac{1}{2}\pi s)} + \frac{1}{3^s\sin(\frac{1}{3}\pi n + \pi s)} + \frac{1}{4^s\sin(\frac{1}{4}\pi n + \frac{3}{2}\pi s)} + \frac{1}{5^s\sin(\frac{1}{5}\pi n + 2\pi s)} + \cdots\right\}.$$

Entry 9.4.2 is the same as (14.4) of Ramanujan's paper [265], [267, p. 193].

On page 272 of [269], Ramanujan offers an asymptotic formula for $r_s(n)$ in three related guises. This asymptotic formula is originally due to Hardy [146], [149, pp. 345–374]. To the best of our knowledge, all the theorems that Ramanujan recorded in his earlier notebooks [268] and lost notebook [269] were discovered by Ramanujan himself, albeit some of his results were rediscoveries. Thus, did Ramanujan, independently of Hardy, establish this asymptotic formula? We know that Ramanujan did discover different asymptotic formulas for $r_s(n)$. To that end, let us recall what Hardy wrote about his and Ramanujan's asymptotic formulas [147, pp. 143, 159]: "I must now introduce ideas which are not to be found (at any rate explicitly) in Ramanujan's work. They are the ideas from which Littlewood and I started in our work on Waring's problem." "Ramanujan formed 'singular series'; thus the series (11.11)–(11.41) of no. 21 of the *Papers* are the singular series relevant in this problem. But his approach to them is quite different; he determines the 'divisor-function' $\delta_{2s}(n)$, as an approximation to $r_{2s}(n)$, independently, and then expands it as a singular series. Here the singular series comes first, and $\delta_{2s}(n)$ appears as its sum." Paper no. 21 is the paper [265], and the series (11.11)–(11.41) are four asymptotic formulas for $\delta_{2s}(n)$, associated with the four residue classes of s modulo 4.

Entry 9.4.3 (p. 272). *If $s \geq 5$, as $n \to \infty$,*

$$r_s(n) = \frac{(\pi n)^{\frac{1}{2}s}}{n\Gamma(\frac{1}{2}s)} \sum_{p,q} \left(e^{-2n\pi ip/q} \left\{ \frac{1}{q} \sum_{\lambda=0}^{q-1} e^{2\pi i\lambda^2 p/q} \right\}^s \right) + O\big(n^{\frac{1}{4}s}\big), \tag{9.4.1}$$

where the outer sum is over all positive integers p and q, with $(p,q)=1$ and $p<q$.

The next entry is an alternative formulation of Entry 9.4.3.

Entry 9.4.4 (p. 272). *Let s be an integer at least 5. For $(p,q)=1$ and $0<p<q$, set*

$$c_{p,q} = \frac{1}{\sqrt{q}} \sum_{\lambda=0}^{q-1} e^{2\pi i\lambda^2 p/q}. \tag{9.4.2}$$

Then, as $n \to \infty$,

$$r_s(n) = \frac{(\pi n)^{\frac{1}{2}s}}{n\Gamma(\frac{1}{2}s)} \sum_{p,q} \left(\frac{e^{-2n\pi ip/q}}{q^{\frac{1}{2}s}} (c_{p,q})^s \right) + O\big(n^{\frac{1}{4}s}\big). \tag{9.4.3}$$

Next, Ramanujan calculates several values of the Gauss sum in (9.4.2), which he uses in his third version of Hardy's asymptotic formula. All of the following values are elementary.

$$c_{1,1}=0; \quad c_{1,2}=0; \quad c_{1,3}=i, \quad c_{2,3}=-i; \quad c_{1,4}=1+i,$$
$$c_{3,4}=1-i; \quad c_{1,5}=1, \quad c_{2,5}=-1, \quad c_{3,5}=-1, \quad c_{4,5}=1;$$
$$c_{1,6}=0, \quad c_{5,6}=0; \quad c_{1,7}=i, \quad c_{2,7}=i, \quad c_{3,7}=-i,$$
$$c_{4,7}=i, \quad c_{5,7}=-i, \quad c_{6,7}=-i; \quad c_{1,8}=1+i, \quad c_{3,8}=-1+i,$$
$$c_{5,8}=-1-i, \quad c_{7,8}=1-i.$$

Entry 9.4.5 (p. 272). *If $s \geq 5$, then, as $n \to \infty$,*

$$r_s(n)=\frac{(\pi n)^{\frac12 s}}{n\Gamma(\frac12 s)}\left\{\frac{1}{1^{\frac12 s}}+\frac{2\cos(\frac12 n\pi-\frac14 s\pi)}{2^{\frac12 s}}+\frac{2\cos(\frac23 n\pi-\frac12 s\pi)}{3^{\frac12 s}}\right.$$
$$+\frac{2\cos(\frac14 n\pi-\frac14 s\pi)+2\cos(\frac34 n\pi-\frac34 s\pi)}{4^{\frac12 s}}$$
$$\left.+\frac{2\cos(\frac25 n\pi)+2\cos(\frac45 n\pi-s\pi)}{5^{\frac12 s}}+\cdots\right\}+O\big(n^{\frac14 s}\big). \tag{9.4.4}$$

Using the calculations prior to Entry 9.4.5, we can easily verify the truth of Entry 9.4.5.

A clear, readable account of Hardy's proof of the asymptotic formulas (9.4.1), (9.4.3), and (9.4.4) can be found in M.I. Knopp's text [183, Chap. 5]. These formulas were extended by P.T. Bateman [23] to include the cases $s = 3, 4$.

9.5 An Aborted Conclusion to [265]?

Page 255 in [269] is devoted to three claims about sums involving $\sigma_k(n)$. Almost certainly, these results were intended to be recorded at the end of the last section, ¶17, of [265]. It does not seem possible to give correct, precise versions of these claims, and so, undoubtedly, Ramanujan, recognizing this, kept this page in his files in the event that he could later find correct renditions.

We discuss the first claim in detail, and show that any precise statement of the claim must incorporate error terms. Then we indicate that even with the addition of error terms, no matter what choices we make for the two parameters, some of the main terms should actually be subsumed in the error terms. Thus, it does not appear possible to state a precise theorem. We provide here a detailed argument by P. Pongsriiam and the second author [63] providing, we hope, conclusive evidence of our claims about Ramanujan's claim. We state the second and third claims without further discussion, since they aim to generalize the first claim.

In the analysis that follows, we make heavy use of the estimate [265, fifth line of Sect. 17], [267, p. 196]

$$\sum_{k=1}^{n} k^s=\zeta(-s)+\frac{(n+\frac12)^{s+1}}{s+1}+O(n^{s-1}), \tag{9.5.1}$$

for any complex number s and positive integer n, where if $s=-1$, we interpret the first two expressions on the right-hand side of (9.5.1) to equal

$$\gamma+\log(n+\tfrac{1}{2}), \tag{9.5.2}$$

where γ denotes Euler's constant. The identity (9.5.1) is well known, but usually not in this form. This elegant formulation is actually found in Chap. 7 of Ramanujan's second notebook [268], [37, p. 150, Entry 1].

We are now ready to record verbatim the first of the three entries on page 255 of [269].

Entry 9.5.1 (p. 255).

$$1^s\sigma_r(1)+2^s\sigma_r(2)+3^s\sigma_r(3)+\cdots+n^s\sigma_r(n) \tag{9.5.3}$$

lies between

$$\zeta(-s)\zeta(-r-s)+\frac{n^{1+s}}{1+s}\zeta(1-r)+\frac{n^{1+r+s}}{1+r+s}\zeta(1+r)\\ +\frac{1}{2}n^s\zeta(-r)+\frac{1}{2}n^{r+s}\zeta(r)+\frac{n^{s+(r+1)/2}}{1-r^2} \tag{9.5.4}$$

and

$$\zeta(-s)\zeta(-r-s)+\frac{n^{1+s}}{1+s}\zeta(1-r)+\frac{n^{1+r+s}}{1+r+s}\zeta(1+r) \tag{9.5.5}\\ +\frac{1}{2}n^s\left\{2\zeta(1-r)-\zeta(-r)\right\}+\frac{1}{2}n^{r+s}\left\{2\zeta(1+r)-\zeta(r)\right\}-\frac{n^{s+(r+1)/2}}{1-r^2}.$$

Note that, perhaps surprisingly, the bounds (9.5.4) and (9.5.5) are asymmetric. Because Ramanujan stated (9.5.4) and (9.5.5) as inequalities, we assume in the analysis that follows that s is real. However, all of our analysis can be extended to encompass complex values of s. In analyzing the sum in Entry 9.5.1, error terms arrive in our estimates. So that these error terms will be $o(1)$ as $n\to\infty$, we furthermore require the hypotheses

$$s+\tfrac{1}{2}r<0, \quad s+r<1, \quad \text{and} \quad s<1. \tag{9.5.6}$$

If we add some additional hypotheses and assume that n is sufficiently large, then it will be shown that (9.5.4) is valid. But, as our analysis shows, (9.5.5) does not appear to be correct, because of the appearances of two extra expressions in (9.5.5). In view of all these observations and restrictions, after readers gain an appreciation of the aforementioned pitfalls, we offer a more precise version of Entry 9.5.1 at the close of this section.

Proof. Set

$$S(s,r):=\sum_{k=1}^{n}k^s\sigma_r(k)=\sum_{k=1}^{n}k^s\sum_{d|k}d^r=\sum_{dl\le n}(dl)^sd^r=\sum_{dl\le n}d^{s+r}l^s. \tag{9.5.7}$$

Applying Dirichlet's hyperbola method to the sum on the far right side of (9.5.7), we find that

$$S(s,r) = \sum_{d\le\sqrt{n}} d^{s+r} \sum_{l\le n/d} l^s + \sum_{l\le\sqrt{n}} l^s \sum_{d\le n/l} d^{s+r} - \sum_{l\le\sqrt{n}} l^s \sum_{d\le\sqrt{n}} d^{s+r}$$
$$= a_1 + a_2 - a_3, \tag{9.5.8}$$

say. We first examine a_1.

If $\{x\}$ denotes the fractional part of x, write

$$\left\lfloor \frac{n}{d} \right\rfloor + \frac{1}{2} = \frac{n}{d} + \frac{1}{2} - \left\{\frac{n}{d}\right\} = \frac{n}{d} + \varepsilon_{n,d},$$

so that

$$\varepsilon_{n,d} = \frac{1}{2} - \left\{\frac{n}{d}\right\} \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Then, applying (9.5.1), we find that

$$a_1 = \sum_{d\le\sqrt{n}} d^{s+r} \left(\frac{\left(\left\lfloor \frac{n}{d}\right\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} + \zeta(-s) + O\left(\left(\frac{n}{d}\right)^{s-1}\right)\right)$$
$$= \sum_{d\le\sqrt{n}} d^{s+r} \left(\frac{\left(\frac{n}{d} + \varepsilon_{n,d}\right)^{s+1}}{s+1} + \zeta(-s) + O\left(\left(\frac{n}{d}\right)^{s-1}\right)\right)$$
$$= \sum_{d\le\sqrt{n}} d^{s+r} \left(\frac{\left(\frac{n}{d}\right)^{s+1}}{s+1} + \left(\frac{n}{d}\right)^{s} \varepsilon_{n,d} + \zeta(-s) + O\left(\left(\frac{n}{d}\right)^{s-1}\right)\right)$$
$$= \frac{n^{s+1}}{s+1} \sum_{d\le\sqrt{n}} d^{r-1} + n^s \sum_{d\le\sqrt{n}} \varepsilon_{n,d} d^r + \zeta(-s) \sum_{d\le\sqrt{n}} d^{s+r}$$
$$+ O\left(n^{s-1} \sum_{d\le\sqrt{n}} d^{r+1}\right)$$
$$= \frac{n^{s+1}}{s+1} \left(\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{r}}{r} + \zeta(1-r) + O\left(n^{\frac{1}{2}(r-2)}\right)\right) + n^s \sum_{d\le\sqrt{n}} \varepsilon_{n,d} d^r$$
$$+ \zeta(-s) \left(\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1} + \zeta(-s-r) + O\left(n^{\frac{1}{2}(s+r-1)}\right)\right)$$

$$
\begin{aligned}
&+ O\left(n^{s-1} \sum_{d\le\sqrt{n}} d^{r+1}\right) \\
&= \frac{n^{s+1}}{s+1} \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^r}{r} + \frac{n^{s+1}}{s+1}\zeta(1-r) + n^s \sum_{d\le\sqrt{n}} \varepsilon_{n,d} d^r \\
&+ \zeta(-s)\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1} + \zeta(-s)\zeta(-s-r) \\
&+ O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s+r-1)}\right) + O\left(n^{s-1}\log n\right), \qquad (9.5.9)
\end{aligned}
$$

where the "extra" factor of $\log n$ arises from the possibility that r may be equal to -2, whence the need to use (9.5.2) in our estimate.

Next, by a similar argument with the use of (9.5.1), or by a change of variables $(s,r) \mapsto (s+r,-r)$,

$$
\begin{aligned}
a_2 &= -\frac{n^{s+r+1}}{s+r+1} \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{-r}}{r} + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) + n^{s+r} \sum_{l\le\sqrt{n}} \frac{\varepsilon_{n,l}}{l^r} \\
&+ \zeta(-s-r)\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} + \zeta(-s-r)\zeta(-s) \\
&+ O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s-1)}\right) + O\left(n^{s+r-1}\log n\right). \qquad (9.5.10)
\end{aligned}
$$

Lastly, by (9.5.1),

$$
\begin{aligned}
a_3 &= \left(\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} + \zeta(-s) + O\left(n^{\frac{1}{2}(s-1)}\right)\right) \\
&\times \left(\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1} + \zeta(-s-r) + O\left(n^{\frac{1}{2}(s+r-1)}\right)\right) \\
&= \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1} + \zeta(-s-r)\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} \\
&+ \zeta(-s)\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1} + \zeta(-s)\zeta(-s-r)
\end{aligned}
$$

$$+ O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s+r-1)}\right) + O\left(n^{\frac{1}{2}(s-1)}\right). \tag{9.5.11}$$

If we now amalgamate (9.5.9)–(9.5.11), simplify, and use our hypotheses (9.5.6), we arrive at

$$\begin{aligned} a_1 + a_2 - a_3 = \zeta(-s)\zeta(-s-r) &+ \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) \\ &+ A_1 + A_2 + A_3 + A_4 - A_5 \\ &+ O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s-1)}\right) + O\left(n^{\frac{1}{2}(s+r-1)}\right), \end{aligned} \tag{9.5.12}$$

where

$$A_1 := n^{s+r} \sum_{l \le \sqrt{n}} \frac{\varepsilon_{n,l}}{l^r}, \quad A_2 := n^s \sum_{d \le \sqrt{n}} \varepsilon_{n,d} d^r, \tag{9.5.13}$$

$$A_3 := \frac{n^{s+1}}{r(s+1)} \left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^r, \quad A_4 := -\frac{n^{s+r+1}}{r(s+r+1)} \left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{-r}, \tag{9.5.14}$$

$$A_5 := \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+1}}{s+1} \frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{s+r+1}}{s+r+1}. \tag{9.5.15}$$

We now turn to A_1, which is defined in (9.5.13). Set

$$E_1 := n^{s+r} \sum_{l \le \sqrt{n}} \left\{\frac{n}{l}\right\} \frac{1}{l^r}. \tag{9.5.16}$$

Then, by (9.5.1),

$$\begin{aligned} A_1 &= \frac{n^{s+r}}{2} \sum_{l \le \sqrt{n}} \frac{1}{l^r} - n^{s+r} \sum_{l \le \sqrt{n}} \left\{\frac{n}{l}\right\} \frac{1}{l^r} \\ &= \frac{n^{s+r}}{2} \sum_{l \le \sqrt{n}} \frac{1}{l^r} - E_1 \\ &= \frac{n^{s+r}}{2} \left(\frac{\left(\lfloor\sqrt{n}\rfloor + \frac{1}{2}\right)^{-r+1}}{-r+1} + \zeta(r) + O\left(n^{-\frac{1}{2}(r+1)}\right) \right) - E_1 \\ &= \frac{n^{s+r}}{2} \left(\frac{n^{\frac{1}{2}(-r+1)}}{-r+1} + \zeta(r) + O\left(n^{-\frac{1}{2}r}\right) \right) - E_1 \\ &= \frac{n^{s+\frac{1}{2}(r+1)}}{2(-r+1)} + \frac{n^{s+r}}{2}\zeta(r) + O\left(n^{s+\frac{1}{2}r}\right) - E_1. \end{aligned} \tag{9.5.17}$$

Next, we examine A_2, also defined in (9.5.13). Put

$$E_2 := n^s \sum_{d \le \sqrt{n}} \left\{ \frac{n}{d} \right\} d^r. \tag{9.5.18}$$

By an argument much like that above, or by invoking the substitution $(s, r) \mapsto (s + r, -r)$, we find that

$$A_2 = \frac{n^{s+\frac{1}{2}(r+1)}}{2(r+1)} + \frac{n^s}{2}\zeta(-r) + O\left(n^{s+\frac{1}{2}r}\right) - E_2. \tag{9.5.19}$$

Combining (9.5.17) and (9.5.19), we arrive at

$$\begin{aligned} A_1 + A_2 &= n^{s+\frac{1}{2}(r+1)} \left(\frac{1}{2(r+1)} + \frac{1}{2(-r+1)} \right) \\ &\quad + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r) + O\left(n^{s+\frac{1}{2}r}\right) - E_1 - E_2 \\ &= \frac{n^{s+\frac{1}{2}(r+1)}}{1 - r^2} + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r) - E_1 - E_2 + O\left(n^{s+\frac{1}{2}r}\right). \end{aligned} \tag{9.5.20}$$

We next turn to the pair A_3, A_4, defined in (9.5.14). Setting

$$\lfloor \sqrt{n} \rfloor + \frac{1}{2} = \sqrt{n} + \varepsilon_n, \quad \text{so that} \quad \varepsilon_n \in \left(-\frac{1}{2}, \frac{1}{2} \right],$$

and invoking the binomial theorem, we find that

$$A_3 = \frac{n^{s+\frac{1}{2}r+1}}{r(s+1)} + \frac{n^{s+\frac{1}{2}(r+1)}}{s+1}\varepsilon_n + O\left(n^{s+\frac{1}{2}r}\right)$$

and

$$A_4 = -\frac{n^{s+\frac{1}{2}r+1}}{r(s+r+1)} + \frac{n^{s+\frac{1}{2}(r+1)}}{s+r+1}\varepsilon_n + O\left(n^{s+\frac{1}{2}r}\right).$$

Recalling that A_5 is defined in (9.5.15), we find that

$$\begin{aligned} A_5 &= \frac{\left(\sqrt{n} + \varepsilon_n\right)^{2s+r+2}}{(s+1)(s+r+1)} \\ &= \frac{n^{s+\frac{1}{2}r+1}}{(s+1)(s+r+1)} + \frac{(2s+r+2)n^{s+\frac{1}{2}(r+1)}\varepsilon_n}{(s+1)(s+r+1)} + O\left(n^{s+\frac{1}{2}r}\right). \end{aligned}$$

The last three calculations thus yield

$$A_3 + A_4 - A_5 = n^{s+\frac{1}{2}r+1} \left(\frac{1}{r(s+1)} - \frac{1}{r(s+r+1)} - \frac{1}{(s+1)(s+r+1)} \right)$$

$$
\begin{aligned}
&+ n^{s+\frac{1}{2}r+\frac{1}{2}}\left(\frac{\varepsilon_n}{s+1}+\frac{\varepsilon_n}{s+r+1}-\frac{(2s+r+2)\varepsilon_n}{(s+1)(s+r+1)}\right)\\
&+ O\left(n^{s+\frac{1}{2}r}\right)\\
= {}& 0 + O\left(n^{s+\frac{1}{2}r}\right). \qquad (9.5.21)
\end{aligned}
$$

Observing that $E_1, E_2 > 0$, we collect (9.5.12), (9.5.20), and (9.5.21) to conclude that

$$
\begin{aligned}
S(s,r) = a_1 + a_2 - a_3 &\leq \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1)\\
&\quad + \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r) - E_1 - E_2\\
&\quad + O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s-1)}\right) + O\left(n^{\frac{1}{2}(s+r-1)}\right)\\
&= \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1)\\
&\quad + \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r)\\
&\quad - E_1 - E_2 + o(1), \qquad (9.5.22)
\end{aligned}
$$

as $n \to \infty$, where we used (9.5.6). If we can show that either E_1 or E_2 is $\gg 1$ as n tends to ∞ for certain values of s and r, then Ramanujan's upper bound (9.5.4) will indeed have been verified.

To that end, and to also obtain a lower bound, we need to estimate E_1 and E_2, given, respectively, in (9.5.16) and (9.5.18). By (9.5.1),

$$
\begin{aligned}
E_1 \leq n^{s+r}\sum_{\ell \leq \sqrt{n}} \frac{1}{\ell^r} &= n^{s+r}\left\{\frac{(\lfloor\sqrt{n}\rfloor + \frac{1}{2})^{-r+1}}{1-r} + \zeta(r) + O\big(n^{-\frac{1}{2}r}\big)\right\}\\
&= \frac{n^{s+\frac{1}{2}(r+1)}}{1-r} + n^{s+r}\zeta(r) + O\big(n^{s+\frac{1}{2}r}\big) \qquad (9.5.23)
\end{aligned}
$$

and

$$
\begin{aligned}
E_2 \leq n^s \sum_{d\leq\sqrt{n}} d^r &= n^s\left\{\frac{(\lfloor\sqrt{n}\rfloor + \frac{1}{2})^{r+1}}{r+1} + \zeta(-r) + O\big(n^{\frac{1}{2}(r-1)}\big)\right\}\\
&= \frac{n^{s+\frac{1}{2}(r+1)}}{r+1} + n^s\zeta(-r) + O\big(n^{s+\frac{1}{2}r}\big). \qquad (9.5.24)
\end{aligned}
$$

We now observe that if we require that $s > 0$ or that $s + r > 0$, then (9.5.22) will imply the truth of (9.5.4).

Returning to our goal of obtaining a lower bound, by (9.5.12), (9.5.20), (9.5.21), (9.5.23), (9.5.24), and (9.5.6), we conclude that

$$\begin{aligned}
S(s,r) = a_1 + a_2 - a_3 \geq \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) \\
- \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r) - n^s\zeta(-r) - n^{s+r}\zeta(r) \\
+ O\left(n^{s+\frac{1}{2}r}\right) + O\left(n^{\frac{1}{2}(s-1)}\right) + O\left(n^{\frac{1}{2}(s+r-1)}\right) \\
= \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) \\
- \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} - \frac{n^s}{2}\zeta(-r) - \frac{n^{s+r}}{2}\zeta(r) + o(1), \qquad (9.5.25)
\end{aligned}$$

which does not agree with (9.5.5), because Ramanujan records additive factors of $n^s\zeta(1-r)$ and $n^{r+s}\zeta(1+r)$, which do not appear in our lower bound above. Observe that in contrast to obtaining the upper bound (9.5.4) when either $s > 0$ or $s + r > 0$, we cannot dispense with the term $o(1)$ in (9.5.25). □

Let us now collect (9.5.22) and (9.5.25) so that we may state an improved version of Ramanujan's Entry 9.5.1.

Entry 9.5.2 (p. 255). *Let s and r be real numbers satisfying the inequalities (9.5.6). Then, for n sufficiently large,*

$$\begin{aligned}
S(s,r) = \sum_{k=1}^{n} k^s\sigma_r(k) \leq \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) \\
+ \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} + \frac{n^s}{2}\zeta(-r) + \frac{n^{s+r}}{2}\zeta(r),
\end{aligned}$$

provided that either $s > 0$ or $s + r > 0$. Furthermore, if n is sufficiently large,

$$\begin{aligned}
S(s,r) \geq \zeta(-s)\zeta(-s-r) + \frac{n^{s+1}}{s+1}\zeta(1-r) + \frac{n^{s+r+1}}{s+r+1}\zeta(r+1) \\
- \frac{n^{s+\frac{1}{2}(r+1)}}{1-r^2} - \frac{n^s}{2}\zeta(-r) - \frac{n^{s+r}}{2}\zeta(r) + o(1).
\end{aligned}$$

We close this section with the remaining two formulas on page 255.

Entry 9.5.3 (p. 255). *For unspecified parameters r and s and $t = [\sqrt{n}]$,*

$$n^{\alpha}\sum_{k=1}^{n} k^{s-\alpha}\sigma_r(k) - \sum_{k=1}^{n} k^s\sigma_r(k)$$

$$= n^{\alpha}\zeta(\alpha - s)\zeta(\alpha - r - s) - \zeta(-s)\zeta(-r - s)$$
$$+ \frac{\alpha n^{1+s}}{(1+s)(1-\alpha+s)}\zeta(1-r) + \frac{\alpha n^{1+r+s}}{(1+r+s)(1-\alpha+r+s)}\zeta(1+r)$$
$$- \frac{\alpha}{2}n^{s-1}\sum_{m=1}^{t}\left(\frac{1}{6} - \epsilon_m + \epsilon_m^2\right)\left(m^{r+1} + \frac{n^r}{m^{r-1}}\right) + O(\ \ ),$$

which lies between

$$n^{\alpha}\zeta(\alpha - s)\zeta(\alpha - r - s) - \zeta(-s)\zeta(-r - s)$$
$$+ \frac{\alpha n^{1+s}}{(1+s)(1-\alpha+s)}\zeta(1-r) + \frac{\alpha n^{1+r+s}}{(1+r+s)(1-\alpha+r+s)}\zeta(1+r)$$
$$- \frac{\alpha}{12}n^{s-1}\zeta(-r-1) - \frac{\alpha}{12}n^{r+s-1}\zeta(r-1) - \frac{\alpha n^{s+r/2}}{3(4-r^2)}$$

and

$$n^{\alpha}\zeta(\alpha - s)\zeta(\alpha - r - s) - \zeta(-s)\zeta(-r - s)$$
$$+ \frac{\alpha n^{1+s}}{(1+s)(1-\alpha+s)}\zeta(1-r) + \frac{\alpha n^{1+r+s}}{(1+r+s)(1-\alpha+r+s)}\zeta(1+r)$$
$$- \frac{\alpha}{24}n^{s-1}\left\{3(1-2^{r-1})\zeta(1-r) - \zeta(-r-1)\right\}$$
$$- \frac{\alpha}{24}n^{r+s-1}\left\{3\left(1 - \frac{1}{2^{r+1}}\right)\zeta(1+r) - \zeta(r-1)\right\} + \frac{\alpha n^{s+r/2}}{6(4-r^2)}.$$

Ramanujan does not indicate the function within the O-term above.

9.6 An Elementary Manuscript on the Divisor Function $d(n)$

Pages 278 and 279 are devoted to a brief manuscript on $d(n)$. All of the arguments are straightforward and familiar to those who have had an introductory course in analytic number theory. We shall therefore simply copy Ramanujan's manuscript while interjecting in square brackets a few comments for readers.

(For $\operatorname{Re} s > 1$), as $s \to 1$,

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{s-1} + \gamma + o(1),$$
$$\sum_{n=1}^{\infty}\frac{\log n}{n^s} = \frac{1}{(s-1)^2} + K + o(1).$$

Hence

$$\frac{\sum_{n=1}^{\infty} \dfrac{\log n}{n^s}}{\sum_{n=1}^{\infty} \dfrac{1}{n^s}} = \frac{1}{s-1} - \gamma + o(1).$$

But

$$\frac{\sum_{n=1}^{\infty} \dfrac{\log n}{n^s}}{\sum_{n=1}^{\infty} \dfrac{1}{n^s}} \equiv \sum_p \frac{\log p}{p^s - 1},$$

(where the sum is over all primes p, and where Ramanujan is using the representation of $\zeta(s)$ as an Euler product.) Therefore

$$\sum_p \frac{\log p}{p^s - 1} = \frac{1}{s-1} - \gamma + o(1)$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_p \frac{\log p}{p^s - 1} = 2\gamma + o(1). \tag{9.6.1}$$

But

$$\sum_{n<x} \frac{1}{n} = \log x + \gamma + o(1). \tag{9.6.2}$$

Assuming $\pi(x) \sim x/\log x$ we have [18, p. 117, Exercise 7]

$$\sum_{p<x} \frac{\log p}{p-1} = \log x + B + o(1). \tag{9.6.3}$$

Hence from (9.6.1)–(9.6.3)

$$\gamma - B = 2\gamma$$

that is

$$B = -\gamma.$$

Hence we have

$$\sum_{p<x} \frac{\log p}{p-1} = \log x - \gamma + o(1).$$

(The well-known asymptotic formula given above is the last formula on page 278. On page 279, Ramanujan begins afresh with a new numbering system for the tagged equations, and so it is not clear whether Ramanujan had intended these two pages to be parts of the same manuscript.)

$$\sum_{n<x} \frac{\log n}{n} = \frac{1}{2}\log^2 x - \gamma_1 + o(1), \tag{9.6.4}$$

where $\gamma_1 = 0.072815845483680\ldots$. ($\gamma_1$ is called the first Stieltjes constant. The Stieltjes constants are examined in detail in Chap. 7 of Ramanujan's second notebook [268], and an extensive discussion of them can be found in [37, pp. 164–165]. Also consult S. Finch's book [119, p. 166–169] for a brief discussion of Stieltjes constants. Because Ramanujan recorded 15 decimal places of γ_1 below (9.6.4), it is likely that he had seen a table of Stieltjes constants to 16 decimal places composed by J.P. Gram [127] in 1895.) As $s \to 1$

$$\sum_{n=1}^{\infty} \frac{\log n}{n^s} = \frac{1}{(s-1)^2} - \gamma_1 + o(1). \tag{9.6.5}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + o(s-1), \tag{9.6.6}$$

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^2 = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \gamma^2 + 2\gamma_1 + o(1) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \tag{9.6.7}$$

where $d(n)$ is the number of divisors of n. But from (9.6.5) (and (9.6.6)) we have

$$\sum_{n=1}^{\infty} \frac{2\gamma + \log n}{n^s} = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + 2\gamma^2 - \gamma_1 + o(1). \tag{9.6.8}$$

Hence we have

$$\sum_{n=1}^{\infty} \frac{2\gamma + \log n}{n^s} - \sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \gamma^2 - 3\gamma_1 + o(1). \tag{9.6.9}$$

But from ((9.6.2) and) (9.6.4) we have

$$\sum_{n<x} \frac{2\gamma + \log n}{n} = \frac{1}{2}\log^2 x + 2\gamma \log x + 2\gamma^2 - \gamma_1 + o(1). \tag{9.6.10}$$

Assuming

$$\sum_{n<x} d(n) = x(2\gamma - 1 + \log x) + O(\sqrt{x}),$$

(then, by partial summation,)

$$\sum_{n<x} \frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma \log x + B + o(1). \tag{9.6.11}$$

Hence from (9.6.9)–(9.6.11),

$$2\gamma^2 - \gamma_1 - B = \gamma^2 - 3\gamma_1.$$

Hence,

$$\sum_{n<x} \frac{d(n)}{n} = (\gamma + \log x)^2 - \frac{1}{2}\log^2 x + 2\gamma_1 + o(1).$$

9.7 Thoughts on the Dirichlet Divisor Problem

Page 368 is an isolated page in [269] on sums involving the divisor function $d(n)$, the number of positive divisors of the positive integer n. It appears that the claims on this page were motivated by the famous Dirichlet divisor problem [145], [150, pp. 268–292], which we now briefly describe. If $x > 0$ and γ denotes Euler's constant, write

$$D(x) := \sum_{n\leq x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x), \tag{9.7.1}$$

where $\Delta(x)$ denotes the "error term." The prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}d(x)$ is counted. Finding the order of magnitude of $\Delta(x)$ for large x is known as *Dirichlet's divisor problem*. It is conjectured that for each $\epsilon > 0$, $\Delta(x) = O(x^{1/4+\epsilon})$, as $x \to \infty$. It was shown by Gauss that $\Delta(x) = O(\sqrt{x})$. For our purposes, it will suffice to use M.G. Voronoï's [310] upper bound, namely,

$$\Delta(x) = O(x^{1/3}\log x). \tag{9.7.2}$$

For an early history of the Dirichlet divisor problem, consult Hardy's paper [145], [150, pp. 268–292], and for a more recent history, consult A. Ivić's book [166, pp. 380–383]. At the bottom of page 368 in [269], one can find a handwritten note by Hardy: "Idea. You can replace the Bessel functions of the Voronoï identity by circular functions, at the price of complicating the 'sum.' Interesting idea, but probably of no value for the study of the divisor problem." In this section, we prove and discuss the claims made by Ramanujan on page 368.

Entry 9.7.1 (p. 368). *If γ denotes Euler's constant, then, as $x \to \infty$,*

$$\sum_{n\leq x} \frac{2\gamma + \log n}{\sqrt{n}} = 2\sqrt{x}\,(\log x + 2\gamma - 2) + C + o(1), \tag{9.7.3}$$

where

$$C = \zeta\left(\frac{1}{2}\right)\left\{\frac{3}{2}\gamma - \frac{1}{4}\pi - \frac{1}{2}\log(8\pi)\right\}. \tag{9.7.4}$$

The value of C in (9.7.4) is actually not given by Ramanujan.

Proof. From Ramanujan's notebooks [268], [37, p. 155] or from his paper [253], [267, pp. 47–49], as $x \to \infty$,

$$\sum_{n\leq x} \frac{2\gamma}{\sqrt{n}} = 2\gamma\left\{2\sqrt{x} + \zeta\left(\frac{1}{2}\right) + O\left(\frac{1}{\sqrt{x}}\right)\right\}. \tag{9.7.5}$$

From [37, p. 226, Entry 24(ii)], as $x \to \infty$,

$$\sum_{n\leq x} \frac{\log n}{\sqrt{n}} = \log x \left\{ 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) + O\left(\frac{1}{\sqrt{x}}\right) \right\} - 4\sqrt{x}$$
$$- \zeta\left(\frac{1}{2}\right)\log x - \zeta\left(\frac{1}{2}\right)\left\{\frac{1}{2}\gamma + \frac{1}{4}\pi + \frac{1}{2}\log(8\pi)\right\} + o(1)$$
$$= 2\sqrt{x}\log x - 4\sqrt{x} - \zeta\left(\frac{1}{2}\right)\left\{\frac{1}{2}\gamma + \frac{1}{4}\pi + \frac{1}{2}\log(8\pi)\right\} + o(1). \tag{9.7.6}$$

If we add (9.7.5) and (9.7.6), we obtain (9.7.3) to complete the proof. We note in passing that in 1899, M. Lerch [214] derived a formula (in closed form) for

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{\sqrt{n}}.$$

□

Entry 9.7.2 (p. 368). *Let*

$$d'(n) = d(n) - 2\gamma - \log n. \tag{9.7.7}$$

Then

$$\sum_{n=1}^{\infty} \frac{d'(n)}{\sqrt{n}} = \zeta^2\left(\frac{1}{2}\right) - C, \tag{9.7.8}$$

where C is defined by (9.7.4).

Proof. We first show that the series on the left-hand side of (9.7.8) converges. Let

$$\mathfrak{D}(x) := \sum_{n\leq x} \{d(n) - 2\gamma - \log n\}.$$

Recall Stirling's formula [11, p. 20, Theorem 1.4.2]

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + O(1), \tag{9.7.9}$$

as x tends to ∞. Using (9.7.1) and (9.7.9), we easily deduce that

$$\mathfrak{D}(x) = \Delta(x) + O(\log x), \tag{9.7.10}$$

as $x \to \infty$. By partial summation,

$$\sum_{n\leq x} \frac{d'(n)}{\sqrt{n}} = \frac{\mathfrak{D}(x)}{\sqrt{x}} - 1 + 2\gamma + \frac{1}{2}\int_1^x \frac{\mathfrak{D}(t)}{t^{3/2}}dt. \tag{9.7.11}$$

By (9.7.2), the first expression on the right-hand side of (9.7.11) tends to 0 as x tends to ∞, and the integral on the right-hand side of (9.7.11) converges absolutely as $x \to \infty$.

We now proceed along the lines of Hardy in [145, Sect. II], [150, pp. 272–273]. Recall the inverse Mellin transform

$$e^{-s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z) s^{-z} dz, \qquad c > 0,\ s > 0. \tag{9.7.12}$$

Upon inverting the order of integration and summation, we easily find that for $c > 1$,

$$\begin{aligned}\mathfrak{F}(s) &:= \sum_{n=1}^{\infty} \frac{d(n) - 2\gamma - \log n}{\sqrt{n}} e^{-s\sqrt{n}} \qquad (9.7.13)\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\left\{\zeta^2(\tfrac{1}{2}+\tfrac{1}{2}z) - 2\gamma\zeta(\tfrac{1}{2}+\tfrac{1}{2}z) + 2\frac{d}{dz}\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\right\} s^{-z} dz.\end{aligned}$$

We now move the line of integration to $(-p - \frac{1}{2} - i\infty, -p - \frac{1}{2} + i\infty)$ by integrating around the rectangle with vertices $c \pm iT, -p - \frac{1}{2} \pm iT$, where p is a positive integer and $T > 0$. Using Stirling's formula [11, p. 21, Corollary 1.1.4]

$$\Gamma(x+iy) \sim \sqrt{2\pi}|y|^{x-1/2}e^{-\pi|y|/2}, \qquad |y| \to \infty, \tag{9.7.14}$$

we easily can show that the integrals along the horizontal sides of the rectangle tend to 0 as $T \to \infty$. If R_α denotes the residue of the integrand's pole at $z = \alpha$, then by the residue theorem,

$$\mathfrak{F}(s) = J_p + R_1 + \sum_{n=0}^{p} R_{-n}, \tag{9.7.15}$$

where

$$J_p := \int_{-p-\frac{1}{2}-i\infty}^{-p-\frac{1}{2}+i\infty} \Gamma(z)\left\{\zeta^2(\tfrac{1}{2}+\tfrac{1}{2}z) - 2\gamma\zeta(\tfrac{1}{2}+\tfrac{1}{2}z) + 2\frac{d}{dz}\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\right\} s^{-z} dz. \tag{9.7.16}$$

To calculate the residue at $z = 1$, we need the Laurent expansions [11, p. 13, Theorem 1.2.5; p. 17]

$$\begin{aligned} s^{-z} &= \frac{1}{s}\left(1 - (\log s)(z-1) + \cdots\right),\\ \Gamma(z) &= 1 - \gamma(z-1) + \cdots,\\ \zeta(\tfrac{1}{2}+\tfrac{1}{2}z) &= \frac{2}{z-1} + \gamma + \cdots, \end{aligned} \tag{9.7.17}$$

as well as the Laurent expansions of $\zeta^2(\frac{1}{2}+\frac{1}{2}z)$ and $\frac{d}{dz}\zeta(\frac{1}{2}+\frac{1}{2}z)$, which are easily obtained from (9.7.17). Omitting the lengthy but straightforward calculation, we find that

$$R_1 = 0, \tag{9.7.18}$$

i.e., the singularity at $z=1$ is removable. The residue at $z=0$ is more easily calculable, and we readily find that

$$R_0 = \zeta^2\left(\frac{1}{2}\right) - 2\gamma\zeta\left(\frac{1}{2}\right) + \zeta'\left(\frac{1}{2}\right). \tag{9.7.19}$$

Because of the presence of $\zeta'\left(\frac{1}{2}\right)$ in (9.7.19), we would like to obtain a more palatable representation for R_0. In the second author's work on the earlier notebooks of Ramanujan, he used a familiar integral representation for $\zeta(s)$ to show that [37, p. 227]

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \frac{1}{2}\log(8\pi) + \frac{1}{4}\pi + \frac{1}{2}\gamma. \tag{9.7.20}$$

(This formula was also established by Lerch [214].) Perhaps a slightly easier method of calculating $\zeta'\left(\frac{1}{2}\right)$ is to employ the functional equation of $\zeta(z)$ in the form [306, p. 16]

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos(\tfrac{1}{2}\pi z)\Gamma(z)\zeta(z), \tag{9.7.21}$$

and then differentiate it and set $z=\frac{1}{2}$. If we substitute (9.7.20) into (9.7.19), we readily find that

$$R_0 = \zeta^2\left(\frac{1}{2}\right) - \zeta\left(\frac{1}{2}\right)\left\{\frac{3}{2}\gamma - \frac{1}{4}\pi - \frac{1}{2}\log(8\pi)\right\}. \tag{9.7.22}$$

There is no need to calculate R_{-n}, $n \geq 1$. However, we need to show that J_p approaches 0 as p tends to ∞, and that $\lim_{p\to\infty}\sum_{n=1}^{p} R_{-n}$, as a power series in s with no constant term, has a finite radius of convergence. Thus, by (9.7.15), $\mathfrak{F}(s)$ will be represented by an analytic function of s, the value of which at $s=0$ will be given by (9.7.22). Hence, we will then have completed the proof of (9.7.8).

To that end, if $z=-p-\frac{1}{2}+iy$, then by the functional equation (9.7.21),

$$\begin{aligned}
|\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)| &= |\zeta(-\tfrac{1}{2}p+\tfrac{1}{4}+\tfrac{1}{2}iy)| \\
&= |2^{-\frac{1}{2}p+\frac{1}{4}+\frac{1}{2}iy}\pi^{-\frac{1}{2}p-\frac{3}{4}+\frac{1}{2}iy}\sin\left\{\tfrac{1}{2}\pi\left(-\tfrac{1}{2}p+\tfrac{1}{4}+\tfrac{1}{2}iy\right)\right\} \\
&\quad\times\Gamma\left(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy\right)\zeta\left(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy\right)| \\
&< K(2\pi)^{-\frac{1}{2}p}e^{\frac{1}{4}\pi|y|}|\Gamma\left(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy\right)|,
\end{aligned} \tag{9.7.23}$$

where K is a positive constant, which will not necessarily be the same at each occurrence in the sequel. Utilizing (9.7.23), the reflection formula for the gamma function, and Stirling's formula (9.7.14), we further find that for $z = -p - \frac{1}{2} + iy$,

$$\begin{aligned}|s^{-z}\Gamma(z)\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)| &< Ks^p(2\pi+\tfrac{1}{2})^{-\frac{1}{2}p}e^{\frac{1}{4}\pi|y|} \\ &\quad\times|\Gamma(-p-\tfrac{1}{2}+iy)\Gamma\left(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy\right)| \\ &= K\left(\frac{s}{\sqrt{2\pi}}\right)^p\left|\frac{\pi e^{\frac{1}{4}\pi|y|}\Gamma(\frac{1}{2}p+\frac{3}{4}-\frac{1}{2}iy)}{\sin\left\{\pi(-p-\frac{1}{2}+iy)\right\}\Gamma(p+\frac{3}{2}-iy)}\right| \\ &< K\left(\frac{s}{\sqrt{2\pi}}\right)^p e^{-\frac{3}{4}\pi|y|}\frac{|\frac{1}{2}y|^{\frac{1}{2}p+\frac{1}{4}}e^{-\frac{1}{2}\pi|\frac{1}{2}y|}}{|y|^{p+1}e^{-\frac{1}{2}\pi|y|}} \\ &= K\left(\frac{s}{2\sqrt{\pi}}\right)^p e^{-\frac{1}{2}\pi|y|}|y|^{-\frac{1}{2}p-\frac{3}{4}} \\ &< K\left(\frac{s}{2\sqrt{\pi}}\right)^p e^{-\frac{1}{2}\pi|y|}, \qquad |y|>1. \end{aligned} \tag{9.7.24}$$

Similarly, it can be shown that [145, p. 6], [150, p. 273]

$$|s^{-z}\Gamma(z)\zeta^2(\tfrac{1}{2}+\tfrac{1}{2}z)| < K\left(\frac{s}{4\pi}\right)^p e^{-\frac{1}{2}\pi|y|}. \tag{9.7.25}$$

There remains one further expression in the integrand of (9.7.16) to estimate. From the functional equation (9.7.21),

$$\begin{aligned}-\zeta'(1-z) &= 2^{1-z}\pi^{-z}\cos(\tfrac{1}{2}\pi z)\Gamma(z)\zeta(z) \\ &\quad\times\left\{-\log(2\pi)-\frac{1}{2}\tan(\tfrac{1}{2}\pi z)\pi+\psi(z)+\frac{\zeta'(z)}{\zeta(z)}\right\},\end{aligned} \tag{9.7.26}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Recall another version of Stirling's formula [11, p. 22, Corollary 1.4.5],

$$\psi(z) = \log z + O(1/|z|), \qquad -\pi+\delta \le \arg z \le \pi-\delta, \tag{9.7.27}$$

for each fixed $\delta > 0$, as $|z| \to \infty$. In particular, (9.7.27) implies that

$$\psi(p+\tfrac{3}{2}-iy) = O(\log|y| + \log p), \tag{9.7.28}$$

as both $|y|$ and p tend to ∞. If we now replace z by $1-z$ in (9.7.26), set $z = -p - \frac{1}{2} + iy$, use (9.7.28), and employ the same analysis that we used in (9.7.24), we find that

$$|s^{-z}\Gamma(z)\frac{d}{dz}\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)| < K\left(\frac{s}{2\sqrt{\pi}}\right)^p(\log|y|+\log p)e^{-\frac{1}{2}\pi|y|}. \tag{9.7.29}$$

Hence, employing our estimates (9.7.24), (9.7.25), and (9.7.29) in (9.7.16), we deduce that

$$\begin{aligned}|J_p| &< K\left(\frac{s}{2\sqrt{\pi}}\right)^p \int_{-\infty}^{\infty} (\log|y| + \log p) e^{-\frac{1}{2}\pi|y|} dy \\ &< K\left(\frac{s}{2\sqrt{\pi}}\right)^p \log p = o(1),\end{aligned}$$

as p tends to ∞, provided that $0 < s < 2\sqrt{\pi}$. Hence, we have shown that $\mathfrak{F}(s)$ can be represented by an analytic function for $0 < s < 2\sqrt{\pi}$. More precisely,

$$\mathfrak{F}(s) = \zeta^2\left(\frac{1}{2}\right) - 2\gamma\zeta\left(\frac{1}{2}\right) + \zeta'\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} R_{-n}.$$

By continuity, we conclude that $\mathfrak{F}(0)$ has the value claimed in (9.7.8). □

Entry 9.7.3 (p. 368). *Let C be defined by* (9.7.4). *If w is any positive number, except $2\sqrt{n}$, then,*

$$\begin{aligned}&\sum_{2\sqrt{n}<w} d(n)\left(1 - \frac{w}{\pi\sqrt{n}}\right) + \sum_{2\sqrt{n}>w} d(n)\left\{\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}}\right\} \\ &= \sum_{2\sqrt{n}<w} d(n) - \sum_{2\sqrt{n}<w} d'(n)\frac{w}{\pi\sqrt{n}} + \sum_{2\sqrt{n}>w} d'(n)\left\{\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}}\right\} \\ &\quad - \sum_{2\sqrt{n}<w} \frac{(2\gamma + \log n)w}{\pi\sqrt{n}} + \sum_{2\sqrt{n}>w} (2\gamma + \log n)\left\{\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}}\right\} \\ &= \sum_{2\sqrt{n}<w} d(n) - \frac{w}{\pi}\left\{\zeta^2\left(\frac{1}{2}\right) - C\right\} + \sum_{2\sqrt{n}>w} d'(n)\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} \\ &\quad - \frac{w}{\pi}\sum_{2\sqrt{n}<w} \frac{2\gamma + \log n}{\sqrt{n}} + \sum_{2\sqrt{n}<w} \frac{2\gamma + \log n}{\sqrt{n}}\left\{\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}}\right\}.\end{aligned}$$

Proof. The two equalities above are merely rearrangements of the expressions on the far left-hand side. One needs to use the definition of $d'(n)$ given by (9.7.7) as well as Entry 9.7.2. □

We have reformulated the next (and last) entry on page 368, which is expressed in terms of the far right-hand side of Entry 9.7.4. For simplicity, we have chosen to use the sum on the far left side in Entry 9.7.4.

Entry 9.7.4 (p. 368). *For $w > 0$,*

$$\begin{aligned}&\sum_{2\sqrt{n}<w} d(n)\left(1 - \frac{w}{\pi\sqrt{n}}\right) + \sum_{2\sqrt{n}>w} d(n)\left\{\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}}\right\} + \frac{w}{\pi}\zeta^2\left(\frac{1}{2}\right) \\ &= \frac{1}{2}w^2 + \frac{1}{4} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{d(n)}{n}e^{-2\pi w\sqrt{n}} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{d(n)}{n}\cos(2\pi w\sqrt{n}). \qquad (9.7.30)\end{aligned}$$

Because of poor photocopying in [269], we are uncertain whether we have correctly identified the expression $\cos(2\pi w\sqrt{n})$ in the last sum on the right-hand side above. Our apprehension arises from $\sqrt{n}$; in fact, Ramanujan uses the summation index q, instead of n. With a magnifying glass, one verifies that the square root is present, but we cannot verify that the argument under the square root sign has been correctly determined.

Entry 9.7.4 does not appear to be correct. For example, consider $0 < w \leq 2$. There is no contribution from the first expression on the left-hand side of (9.7.30). Since

$$\frac{2}{\pi}\sin^{-1}\frac{w}{2\sqrt{n}} - \frac{w}{\pi\sqrt{n}} = \frac{w^3}{24\pi n^{3/2}} + \cdots,$$

and since for every $\epsilon > 0$, $d(n) = O(n^{\epsilon})$, as $n \to \infty$, we see that the infinite series on the left-hand side of (9.7.30) converges absolutely and uniformly for $0 \leq w \leq 2$. Of course, the first series on the right-hand side of (9.7.30) converges absolutely and uniformly on $\delta \leq w \leq 2$ for each $\delta > 0$. However, the latter series on the right-hand side of (9.7.30) rapidly oscillates for $0 < w \leq 2$. In fact, it is not clear for which values of w (if any) the series converges.

The series

$$\sum_{n=1}^{\infty}\frac{d(n)}{n}\cos(2\pi wn)$$

has been the subject of several investigations, in terms of both finding criteria for convergence and for estimating its partial sums. In particular, see papers by S.D. Chowla [94], [95, pp. 230–249], A. Walfisz [311], and J.R. Wilton [319].

In his brief note at the bottom of page 368, Hardy refers to the Voronoï summation formula and replacing the Bessel functions by circular functions. If $Y_1(x)$ and $K_1(x)$ are the Bessel functions usually so denoted [314, pp. 64, 78], $x > 0$, and γ denotes Euler's constant, then

$$\begin{aligned}\sum_{n\leq x}{}' d(n) &= x(\log x + 2\gamma - 1) + \frac{1}{4}\\ &\quad - \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2}\left(Y_1(4\pi\sqrt{nx}) + \frac{2}{\pi}K_1(4\pi\sqrt{nx})\right), \qquad (9.7.31)\end{aligned}$$

where the prime $'$ on the summation sign on the left-hand side indicates that if x is an integer N, then only $\frac{1}{2}d(N)$ is counted. See also papers by A. Oppenheim [239], K. Chandrasekharan and R. Narasimhan [90], and Berndt [26] for proofs. In view of the first expression on the left-hand side of (9.7.30), Hardy might also be thinking of a corresponding identity for $\sum_{n\leq x} d(n)(n-x)$, namely,

$$\begin{aligned}\sum_{n\leq x} d(n)(x-n) &= \frac{1}{2}x^2\left(\log x + 2\gamma - \frac{3}{2}\right) + \frac{1}{4}x - \frac{1}{144}\\ &\quad - \frac{1}{2\pi}\sum_{n=1}^{\infty} d(n)\frac{x}{n}\left(Y_2(4\pi\sqrt{nx}) - \frac{2}{\pi}K_2(4\pi\sqrt{nx})\right), \quad (9.7.32)\end{aligned}$$

which was first proved by Oppenheim [239]. See also papers by Chandrasekharan and Narasimhan [90] and Berndt [26]. Hardy's remark on replacing the Bessel functions by circular functions undoubtedly refers to the fact that the asymptotic expansions of both $Y_\nu(x)$ and $K_\nu(x)$, as $x \to \infty$, involve trigonometric functions [314, pp. 199, 202]. However, replacing each of these functions in either (9.7.31) or (9.7.32) by the first terms in their asymptotic expansions would, of course, not yield an identity, which is what Ramanujan claims to be true. It is interesting that inverse trigonometric functions, instead of trigonometric functions, appear in Ramanujan's assertion (9.7.30).

10

Identities Related to the Riemann Zeta Function and Periodic Zeta Functions

10.1 Introduction

On page 196 in his lost notebook, Ramanujan lists several identities that are related to the Riemann zeta function, Dirichlet L-series, and periodic zeta functions. Some of the identities are connected to previous results of Ramanujan in [256] and [258], but none of the identities on page 196 can be found in these papers. Furthermore, all of the identities on page 196 are new. The purpose of this chapter is to examine all of these interesting identities. Two of the identities were examined and generalized in a paper that the second author wrote with H.H. Chan and Y. Tanigawa [47].

10.2 Identities for Series Related to $\zeta(2)$ and $L(1,\chi)$

At the top of page 196 in [269], Ramanujan records three identities related to $\zeta(2)$, and at the bottom of the page, he states a similar result related to $L(1,\chi)$, where χ is the nonprincipal primitive character modulo 4. In each of the first three identities, the coefficient 4 of the series on the right-hand side must be replaced by 2. We record the results in corrected form.

Entry 10.2.1 (p. 196). *Let* $\operatorname{Re} x \geq 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - 2\pi^2 x^{3/2} \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt, \quad (10.2.1)$$

where the principal value of the square root is taken.

Proof. We assume throughout the proof that $x \geq 0$. The more general result for $\operatorname{Re} x \geq 0$ will then hold by analytic continuation. We begin with the familiar theta transformation formula, which is found in Ramanujan's

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notebooks [268], [39, p. 43, Entry 27(i)]. It will be convenient, however, to use the formulation, for $\mathrm{Re}\, t > 0$,

$$\sum_{n=1}^{\infty} e^{-n^2\pi t} = -\frac{1}{2} + \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t}} \sum_{n=1}^{\infty} e^{-n^2\pi/t}, \tag{10.2.2}$$

which is found in Titchmarsh's treatise [306, p. 22, Eq. (2.6.3)], for example. Integrate both sides of (10.2.2) over $[0, x]$, invert the order of integration and summation by absolute convergence, and multiply both sides by $-\pi$ to reach the identity

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^x \frac{e^{-n^2\pi t}}{\sqrt{t}} dt. \tag{10.2.3}$$

In comparing (10.2.3) with (10.2.1), we see that we must address the integrals on the right side of (10.2.3). First, set $t = x/u$ and then set $n^2 u = (n + tx)^2$. Hence,

$$\int_0^x \frac{e^{-n^2\pi t}}{\sqrt{t}} dt = \sqrt{x} \int_1^{\infty} \frac{e^{-n^2\pi u/x}}{u^{3/2}} du = 2x^{3/2} n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt. \tag{10.2.4}$$

When examining (10.2.1) in relation to (10.2.3) and (10.2.4), we see that it remains to show that

$$2\pi \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt = n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt. \tag{10.2.5}$$

Integrating the latter integral by parts, in particular integrating $1/(n + tx)^2$ and differentiating the exponential, we readily find that for $\mathrm{Re}\, x > 0$,

$$n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt = \frac{e^{-n^2\pi/x}}{x} - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt.$$

On the other hand, after a little trickery and then a direct integration, we find that

$$\begin{aligned} 2\pi \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt &= \frac{2\pi}{x} \int_0^{\infty} (n+tx) e^{-\pi(n+tx)^2/x} dt \\ &\quad - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt \\ &= \frac{e^{-n^2\pi/x}}{x} - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt. \end{aligned}$$

From these two calculations, we see that (10.2.5) has been demonstrated, and so the proof is complete. □

In Chap. 15 of his second notebook [268], [38, p. 306, Theorem 3.1], Ramanujan stated a general asymptotic formula for

$$\sum_{n=1}^{\infty} e^{-xn^p} n^{m-1},$$

as $x \to 0^+$. If we set $p = 2$ and $m = -1$, and replace x by πx in this asymptotic formula, we find that

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x + o(1), \tag{10.2.6}$$

as $x \to 0^+$, which should be compared with (10.2.1). In [38, pp. 306–308], a proof of Ramanujan's general asymptotic formula was obtained by contour integration. In the course of this proof, the error term, i.e., $o(1)$ in (10.2.6), is represented by a certain contour integral. It seems to be very difficult, however, to transform this contour integral into the expression involving the infinite series on the right-hand side of (10.2.1).

Entry 10.2.2 (p. 196). *Let* $\operatorname{Re} x \geq 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{\cos(n^2\pi x)}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{\frac{x}{2}} + 2\pi^2 x^{3/2} \times \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-2n\pi t} \cos\left(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi t^2 x\right) dt \tag{10.2.7}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n^2} = \pi\sqrt{\frac{x}{2}} - \frac{1}{2}\pi x + 2\pi^2 x^{3/2} \times \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-2n\pi t} \sin\left(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi t^2 x\right) dt, \tag{10.2.8}$$

where the principal value of the square root is taken.

Proof. As in the previous proof, we shall assume that $x \geq 0$; an appeal to analytic continuation then establishes Entry 10.2.2 for $\operatorname{Re} x \geq 0$. We shall prove (10.2.7) and (10.2.8) with x replaced by y. In (10.2.1), replace x by $z = x + iy$, with $y \geq 0$. Let $\theta = \arg z$. Then (10.2.1) takes the form

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi z}}{n^2} &= \frac{\pi^2}{6} - \pi|z|^{1/2}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta) + \frac{1}{2}\pi(x+iy) \\ &\quad - 2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta) \\ &\quad \times \int_0^{\infty} t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 + 2it(n+tx)y - t^2y^2\right)(x-iy)\right\} dt.\end{aligned} \tag{10.2.9}$$

Now,

$$\begin{aligned}
E(x,y) &:= -2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta)t \\
&\quad\times \exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 + 2it(n+tx)y - t^2y^2\right)(x-iy)\right\} \\
&= -2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta)t\exp\Bigg(-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x \\
&\quad + 2t(n+tx)y^2 + i\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right)\Bigg),
\end{aligned}$$

from which we see that

$$\begin{aligned}
\operatorname{Re} E(x,y) &= -2\pi^2|z|^{3/2}\cos\tfrac{3}{2}\theta\, t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x\right\} \\
&\quad\times\cos\frac{\pi}{|z|^2}\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right) \\
&\quad + 2\pi^2|z|^{3/2}\sin\tfrac{3}{2}\theta\, t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x\right\} \\
&\quad\times\sin\frac{\pi}{|z|^2}\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right).
\end{aligned}$$

Setting $x = 0$ and $\theta = \frac{1}{2}\pi$, we find that

$$\begin{aligned}
\operatorname{Re} E(0,y) &= 2\pi^2y^{3/2}\frac{1}{\sqrt{2}}te^{-2n\pi t}\cos\left(-\frac{\pi n^2}{y} + \pi t^2y\right) \\
&\quad - 2\pi^2y^{3/2}\frac{1}{\sqrt{2}}te^{-2n\pi t}\sin\left(-\frac{\pi n^2}{y} + \pi t^2y\right) \\
&= 2\pi^2y^{3/2}te^{-2n\pi t}\cos\left(\frac{\pi}{4} - \frac{\pi n^2}{y} + \pi t^2y\right).
\end{aligned} \tag{10.2.10}$$

If we now use (10.2.10) in (10.2.9), we deduce (10.2.7) with x replaced by y.

A similar calculation of $\operatorname{Im} E(x,y)$ followed by setting $x = 0$ and $\theta = \frac{1}{2}\pi$ yields (10.2.8) with x replaced by y. □

Entry 10.2.3 (p. 196). *For $x \geq 0$,*

$$\sum_{n=0}^{\infty}(-1)^n\frac{e^{-(2n+1)^2\pi x/4}}{2n+1} = \frac{\pi}{4} - \pi\sqrt{x}\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-\pi(2n+1+2tx)^2/(4x)}dt. \tag{10.2.11}$$

Proof. We begin by specializing the well-known theta relation for an odd primitive character [101, p. 70, Eq. (9)]. In our case, this odd primitive character is the real nonprincipal character modulo 4. Accordingly, for $t > 0$,

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-(2n+1)^2\pi t/4} = t^{-3/2}\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-(2n+1)^2\pi/(4t)}. \tag{10.2.12}$$

Integrate both sides of (10.2.12) over $[0, x]$, and then multiply both sides by $-\pi/4$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n \frac{e^{-(2n+1)^2\pi x/4}}{2n+1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \frac{\pi}{4}\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_0^x t^{-3/2}e^{-(2n+1)^2\pi/(4t)}dt \\ &= \frac{\pi}{4} - \frac{\pi\sqrt{x}}{4}\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_1^{\infty}\frac{e^{-(2n+1)^2\pi u/(4x)}}{\sqrt{u}}du, \qquad (10.2.13)\end{aligned}$$

where we utilized Leibniz's series for $\pi/4$ and made the substitution $t = x/u$ in the integrals on the right side. Next, set $(2n+1)^2u = (2n+1+2tx)^2$. Then

$$\int_1^{\infty}\frac{e^{-(2n+1)^2\pi u/(4x)}}{\sqrt{u}}du = \frac{4x}{2n+1}\int_0^{\infty} e^{-(2n+1+2tx)^2\pi/(4x)}dt. \qquad (10.2.14)$$

If we substitute (10.2.14) into (10.2.13), we obtain (10.2.11) to complete the proof. □

10.3 Analogues of Gauss Sums

We now offer three claims from the middle of page 196 of [269]. These were first proved in a more general setting by Berndt, Chan, and Tanigawa [47]. More precisely, they evaluate the sum

$$\sum_{n=1}^{\infty}\frac{e^{2\pi i n^2/k}}{n^{2m}},$$

where m and k are positive integers, in several ways, obtaining evaluations in terms of trigonometric functions, Stirling numbers of the second kind, and ballot numbers. On page 196, Ramanujan considers only the case $m = 1$.

Entry 10.3.1 (p. 196). *Let a be an even positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{\cos\left(\dfrac{\pi n^2}{a}\right)}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right), \qquad (10.3.1)$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\dfrac{\pi n^2}{a}\right)}{n^2} = -\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right), \qquad (10.3.2)$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\dfrac{\pi}{4}+\dfrac{\pi n^2}{a}\right)}{n^2} = \frac{\pi^2}{6\sqrt{2}} - \frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi r^2}{a}\right). \qquad (10.3.3)$$

We first prove (10.3.3) assuming the truth of (10.3.1) and (10.3.2).

Proof of (10.3.3) *of Entry* 10.3.1. Using the addition formulas for sin and cos, we easily find that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{4}+\frac{\pi n^2}{a}\right)}{n^2}&=\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^2}+\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^2}\right)\\&=\frac{\pi^2}{6\sqrt{2}}-\frac{\pi^2}{\sqrt{2}a}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)+\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)\right\}\\&=\frac{\pi^2}{6\sqrt{2}}-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi r^2}{a}\right).\end{aligned}$$

□

We evaluate the more general series

$$S_a(r):=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^r}\qquad\text{and}\qquad T_a(r):=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^r},\tag{10.3.4}$$

where r is an even positive integer. In order to effect these evaluations, we need to introduce periodic Bernoulli numbers.

Let $A=\{a_n\}$, $-\infty<n<\infty$, denote a sequence of numbers with period k. Then the periodic Bernoulli numbers $B_n(A)$, $n\geq 0$, can be defined [66, p. 55, Proposition 9.1], for $|z|<2\pi/k$, by

$$\frac{z\sum_{n=0}^{k-1}a_ne^{nz}}{e^{kz}-1}=\sum_{n=0}^{\infty}\frac{B_n(A)}{n!}z^n.$$

Furthermore [66, p. 56, Eq. (9.5)], for each positive integer n,

$$B_n(A)=k^{n-1}\sum_{j=0}^{k-1}a_{-j}B_n\left(\frac{j}{k}\right),\tag{10.3.5}$$

where $B_n(x)$, $n\geq 0$, denotes the nth Bernoulli polynomial. We say that A is even if $a_n=a_{-n}$ for every integer n.

The complementary sequence $B=\{b_n\}$, $-\infty<n<\infty$, is defined by [66, p. 32]

$$b_n=\frac{1}{k}\sum_{j=0}^{k-1}a_je^{-2\pi ijn/k}.\tag{10.3.6}$$

It is easily checked that if A is even, then B is even, and that (10.3.6) holds if and only if

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi ijn/k}, \quad -\infty < n < \infty. \tag{10.3.7}$$

Now set

$$\zeta(s; A) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \operatorname{Re} s > 1.$$

If A and r are even and if $r \geq 2$, then [66, p. 49, Eq. (6.25)]

$$\zeta(r; B) = \frac{(-1)^{r+1} B_r(A)}{2\, r!} \left(\frac{2\pi i}{k}\right)^r.$$

From (10.3.6) and (10.3.7), we see that the sequences A and B are not symmetric. Thus, we note from above that since A is even,

$$\zeta(r; A) = \frac{(-1)^{r+1} B_r(B) k}{2\, r!} \left(\frac{2\pi i}{k}\right)^r. \tag{10.3.8}$$

We are now ready to state general evaluations in closed form for $S_a(r)$ and $T_a(r)$.

Theorem 10.3.1. *If $S_a(r)$ and $T_a(r)$ are defined by (10.3.4) and if r and a are even positive integers, then*

$$S_a(r) = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r! \sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \tag{10.3.9}$$

and

$$T_a(r) = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r! \sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right). \tag{10.3.10}$$

In our work below, we need the value of the Gauss sum [54, p. 43, Exercise 5]

$$\sum_{n=0}^{c-1} e^{\pi i n^2/c} = e^{\pi i/4} \sqrt{c}, \tag{10.3.11}$$

where c is an even positive integer.

Before proceeding further, we show that (10.3.1) and (10.3.2) are special cases of (10.3.9) and (10.3.10), respectively. Let $r = 2$ in Theorem 10.3.1. Recall that $B_2(x) = x^2 - x + \frac{1}{6}$. Then

$$\begin{aligned} S_a(2) &= \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} + \frac{1}{6} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\ &= \frac{\pi^2}{6\sqrt{a}} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\ &= \frac{\pi^2}{6} + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right), \end{aligned}$$

upon the use of (10.3.11) twice.

The proof of (10.3.2) follows along the same lines, but note that in this case, by (10.3.11),

$$\sum_{m=0}^{a-1} \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) = 0.$$

Proof of Theorem 10.3.1. Let

$$a_n = \cos\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which is an even periodic sequence with period a, since a is even. Then, from (10.3.6) and (10.3.11),

$$\begin{aligned} b_{-m} &= \frac{1}{a} \sum_{j=0}^{a-1} \cos\left(\frac{\pi j^2}{a}\right) e^{2\pi i j m/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i (j+m)^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i (j+m)^2/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i j^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i j^2/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a + \pi i/4} \sqrt{a} + \frac{1}{2a} e^{\pi i m^2/a - \pi i/4} \sqrt{a} \\ &= \frac{1}{\sqrt{a}} \cos\left(\frac{\pi m^2}{a} - \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{a}} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right). \end{aligned}$$

Therefore, by (10.3.5), with B in place of A,

$$B_n(B) = a^{n-3/2} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) B_n\left(\frac{m}{a}\right). \tag{10.3.12}$$

If we substitute (10.3.12) into (10.3.8) and simplify, we deduce (10.3.9).

The proof of (10.3.10) is analogous to that for (10.3.9). Now we set

$$a_n = \sin\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which of course is even, and repeat the same kind of argument that we gave above. □

We now provide another evaluation of the series on the left-hand sides of (10.3.1) and (10.3.2) in closed form. However, we obtain evaluations in entirely different forms from those claimed by Ramanujan in Entry 10.3.1.

Theorem 10.3.2. *Let a be an even positive integer, $a \geq 2$. Then*

$$S_a(2) = \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right) \tag{10.3.13}$$

and

$$T_a(2) = \frac{\pi^2\sin(a\pi/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\sin\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right). \tag{10.3.14}$$

Proof. Setting $n = ka + j$, $0 \leq k < \infty$, $1 \leq j \leq a$, we find that

$$\begin{aligned} S_a(2) &= \sum_{j=1}^{a}\cos\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(ka+j)^2} \\ &= \frac{\pi^2}{6a^2} + \frac{1}{a^2}\sum_{j=1}^{a-1}\cos\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2}. \end{aligned} \tag{10.3.15}$$

Singling out the term for $j = a/2$ and noting that the terms in the outer sum with indices j and $a - j$ are identical, we find from (10.3.15) that

$$\begin{aligned} S_a(2) &= \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{1}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right) \\ &\quad\times\left(\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2} + \sum_{k=0}^{\infty}\frac{1}{(k+(a-j)/a)^2}\right) \\ &=: \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{1}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right)U(j,a), \end{aligned} \tag{10.3.16}$$

say. There remains the evaluation of $U(j,a)$.

First observe that if for $-\infty < k \leq -1$, we set $k = -r - 1$, then

$$\sum_{k=-\infty}^{\infty}\left(\frac{1}{(k+j/a)^2}+\frac{1}{(k+(a-j)/a)^2}\right)$$
$$=\sum_{k=0}^{\infty}\left(\frac{1}{(k+j/a)^2}+\frac{1}{(k+(a-j)/a)^2}\right)$$
$$+\sum_{r=0}^{\infty}\left(\frac{1}{(-r-1+j/a)^2}+\frac{1}{(-r-j/a)^2}\right)=2U(j,a). \tag{10.3.17}$$

It therefore suffices to evaluate the bilateral sum in (10.3.17).

To evaluate $U(j,a)$, recall the partial fraction decomposition

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}+\frac{1}{z-n}\right).$$

Differentiating once above, we find that

$$\pi^2\csc^2(\pi z)=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}. \tag{10.3.18}$$

Putting $z=r/k$ in (10.3.18), we deduce that

$$U(j,a)=\pi^2\csc^2(\pi r/k). \tag{10.3.19}$$

Putting (10.3.19) in (10.3.16), we complete the proof of (10.3.13).

The proof of (10.3.14) follows along exactly the same lines. In analogy with (10.3.17), we now easily deduce that

$$T_a(2)=\frac{1}{a^2}\sum_{j=1}^{a}\sin\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2}.$$

By the same identical argument that we used above, we conclude that

$$T_a(2)=\frac{\pi^2\sin(a\pi/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\sin\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right).$$

□

We record a few examples to illustrate Theorem 10.3.2, namely,

$$S_2(2)=\frac{\pi^2}{24},\qquad S_4(2)=-\frac{\pi^2}{48}+\frac{\pi^2\sqrt{2}}{16},\qquad S_6(2)=-\frac{\pi^2}{72}+\frac{\pi^2\sqrt{3}}{18},$$
$$T_2(2)=\frac{\pi^2}{8},\qquad T_4(2)=\frac{\pi^2\sqrt{2}}{16},\qquad T_6(2)=\frac{\pi^2}{24}+\frac{\pi^2\sqrt{3}}{54}.$$

Equating the evaluations of $S_a(2)$ and $T_a(2)$ in (10.3.13) and (10.3.14), respectively, with those in (10.3.1) and (10.3.2), we obtain identities that would be surprising if we had not known of their origins, namely,

$$\frac{\pi^2}{6a^2}+\frac{\pi^2\cos(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\cos\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$
$$=\frac{\pi^2}{6}-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)$$

and

$$\frac{\pi^2\sin(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\sin\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$
$$=-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right).$$

Note that on the left-hand sides above, the sums contain only trigonometric functions, while on the right-hand sides the sums contain both polynomials and trigonometric functions. Trigonometric identities involving polynomials in the summands appear to be rare. The sums on both sides of the identities may be regarded as new analogues of Gauss sums.

In fact, Ramanujan states a second equality for the sum on the left side of (10.3.3). We slightly reformulate this result in the next entry.

Entry 10.3.2 (p. 196). *If a is an even positive integer, then*

$$\frac{4\pi^2}{a^{3/2}}\left\{\frac{1}{8\pi}+\sum_{n=1}^{\infty}\frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}\right\}-2^{3/2}\pi^2\left\{\frac{1}{8\pi a}+\sum_{n=1}^{\infty}\frac{n}{e^{2n\pi a}-1}\right\}$$
$$=-\frac{\pi^2}{a^{5/2}}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right). \qquad (10.3.20)$$

Proof. Our proof depends on two results from Ramanujan's papers [256, 262]. First, if a is an even positive integer [262, Eq. (17)], [267, p. 132], then

$$\frac{1}{8\pi}+\sum_{n=1}^{\infty}\frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}=\int_0^{\infty}\frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx+a\sqrt{\tfrac{1}{2}a}\sum_{n=1}^{\infty}\frac{n}{e^{2n\pi a}-1}. \qquad (10.3.21)$$

Now, from [256, Eq. (50)], [267, p. 67],

$$\int_0^{\infty}\frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx=\frac{1}{2}\int_0^{\infty}\frac{\cos(\pi u/a)}{e^{2\pi\sqrt{u}}-1}du$$
$$=\frac{1}{2}\left(\frac{\sqrt{a/2}}{4\pi}-\frac{1}{2a}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right)\right)$$
$$=\frac{\sqrt{a}}{8\pi\sqrt{2}}-\frac{1}{4a}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right). \qquad (10.3.22)$$

If we substitute (10.3.22) in (10.3.21) and then multiply both sides of the resulting equality by $4\pi^2/a^{3/2}$, we deduce that

$$\frac{\pi}{2a^{3/2}} + \frac{4\pi^2}{a^{3/2}} \sum_{n=1}^{\infty} \frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}$$
$$= \frac{\pi}{2\sqrt{2}a} - \frac{\pi^2}{a^{5/2}} \sum_{r=1}^{a} r(a-r)\cos\left(\frac{\pi r^2}{a}\right) + \frac{4\pi^2}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{n}{e^{2n\pi a}-1},$$

which is easily seen to be equivalent to (10.3.20). □

11

Two Partial Unpublished Manuscripts on Sums Involving Primes

11.1 Introduction

Two unpublished manuscripts by Ramanujan on sums involving primes, but in the handwriting of G.N. Watson, can be found on pages 228–232 in [269]. The original manuscripts in Ramanujan's handwriting are in the library at Trinity College, Cambridge. The first manuscript contains four sections, while the second contains five. For each of the two papers, we copy Ramanujan's work, and then we supply more details, if needed, and offer further comments after each manuscript. We have taken the liberty of making minor notational adjustments. Several claims in the first manuscript are fallacious, indicating that it emanates from an earlier portion of Ramanujan's career sometime before he departed for England in March 1914.

11.2 Section 1, First Paper

In this paper I consider approximate summations of series involving prime numbers. The approximate summation of the series

$$\phi(2) + \phi(3) + \phi(5) + \cdots + \phi(p) \tag{11.2.1}$$

in terms of the number of terms is somewhat more regular and approximate than that in terms of p. Let p be the greatest prime not exceeding x, $\pi(x)$ the number of primes not exceeding x and also let

$$\vartheta(x) = \log 2 + \log 3 + \log 5 + \cdots + \log p \qquad \text{and } \pi(x) = n.$$

Then it can easily be shown that

$$\pi(x)\log x - \vartheta(x) = \int_2^x \frac{\pi(t)}{t}\,dt. \tag{11.2.2}$$

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Without assuming the prime number theorem we have

$$\log x = \log n + \log\log n + O(1). \tag{11.2.3}$$

It follows that

$$\log 2 + \log 3 + \log 5 + \cdots + \log p = n\log n + n\log\log n + O(n). \tag{11.2.4}$$

This is obtained by purely elementary reasoning. But

$$\vartheta(p) \sim p \tag{11.2.5}$$

is as deep as the prime number theorem.

11.3 Section 2, First Paper

Let us now assume the Riemann hypothesis and express $\vartheta(x)$ in terms of n. We have

$$\vartheta(x) = x - \sqrt{x} - \sum_{\rho} \frac{x^{\rho}}{\rho} + O(x^{1/3}), \tag{11.3.1}$$

where ρ is a complex root of the Zeta-function, and

$$\begin{aligned} \pi(x) &= \mathrm{Li}(x) - \frac{1}{\log x}\left(\sqrt{x} + \sum_{\rho}\frac{x^{\rho}}{\rho}\right) + O\left(\frac{\sqrt{x}}{\log^2 x}\right) \\ &= \mathrm{Li}\left\{x - \sqrt{x} - \sum_{\rho}\frac{x^{\rho}}{\rho} + O\left(\frac{\sqrt{x}}{\log x}\right)\right\} \\ &= \mathrm{Li}\left\{\vartheta(x) + O\left(\frac{\sqrt{x}}{\log x}\right)\right\}, \end{aligned} \tag{11.3.2}$$

where

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}.$$

It follows from (11.3.2) that

$$\vartheta(x) = \mathrm{Li}^{-1}(n) + O\left(\sqrt{\frac{n}{\log n}}\right). \tag{11.3.3}$$

But in terms of p we know only that

$$\vartheta(x) = p + O\left(\sqrt{p}\log^2 p\right). \tag{11.3.4}$$

Let us consider more precisely the error in (11.3.3). We have

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{\log x}\left(\sqrt{x} + \sum_{\rho}\frac{x^{\rho}}{\rho}\right)$$

$$- \frac{1}{\log^2 x}\left(2\sqrt{x} + \sum_\rho \frac{x^\rho}{\rho^2}\right) + O\left(\frac{\sqrt{x}}{\log^2 x}\right)$$

$$= \mathrm{Li}\left\{\vartheta(x) - \frac{1}{\log x}\left(2\sqrt{x} + \sum_\rho \frac{x^\rho}{\rho^2}\right) + O\left(\frac{\sqrt{x}}{\log^2 x}\right)\right\}. \tag{11.3.5}$$

But

$$\left|\sum_\rho \frac{x^\rho}{\rho^2}\right| \le \sum_\rho \left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\sum_\rho \frac{1}{\rho(1-\rho)} = (2+\gamma-\log(4\pi))\sqrt{x}, \tag{11.3.6}$$

where γ is Euler's constant. It follows that

$$\begin{cases} \limsup\left\{\vartheta(x) - \mathrm{Li}^{-1}(n)\right\}\sqrt{\dfrac{\log n}{n}} = 4+\gamma-\log(4\pi) \quad (= 2.046 \text{ approx}) \\ \liminf\left\{\vartheta(x) - \mathrm{Li}^{-1}(n)\right\}\sqrt{\dfrac{\log n}{n}} = \log(4\pi)-\gamma \quad (= 1.954 \text{ approx}). \end{cases} \tag{11.3.7}$$

Thus we see that if the series of the form (11.2.1) are expressed in terms of $\vartheta(x)$, then they can immediately be expressed in terms of n with the help of (11.3.3).

11.4 Section 3, First Paper

It is easy to show that, if $\phi'(x)$ is continuous between 2 and x, then

$$\phi(2)+\phi(3)+\phi(5)+\cdots+\phi(p) = \pi(x)\phi(x) - \int_2^x \phi'(t)\pi(t)dt. \tag{11.4.1}$$

$$\phi(2)\log 2+\phi(3)\log 3+\phi(5)\log 5+\cdots+\phi(p)\log p = \vartheta(x)\phi(x) - \int_2^x \phi'(t)\vartheta(t)dt. \tag{11.4.2}$$

But it is easily seen that

$$\phi(x)\vartheta(x) - \int \phi'(x)\vartheta(x)dx = \int \phi(x)dx - \{x-\vartheta(x)\}\phi(x) + \int \phi'(x)\{x-\vartheta(x)\}dx. \tag{11.4.3}$$

Again we have by Taylor's theorem

$$\int^{\vartheta(x)} \phi(t)dt = \int^x \phi(x)dx - \{x-\vartheta(x)\}\phi(x) + \frac{1}{2}\{x-\vartheta(x)\}^2\phi'\{x(1-\theta)+\theta\vartheta(x)\}, \tag{11.4.4}$$

where $0 \le \theta \le 1$. It follows from (11.4.2)–(11.4.4) that

$$\phi(2)\log 2+\phi(3)\log 3+\phi(5)\log 5+\cdots+\phi(p)\log p = C \tag{11.4.5}$$

$$+ \int^{\vartheta(x)} \phi(t)dt + \int \phi'(x)\{x-\vartheta(x)\}dx - \frac{1}{2}\{x-\vartheta(x)\}^2\phi'\{x(1-\theta)+\theta\vartheta(x)\}.$$

11.5 Section 4, First Paper

Now let us consider the two forms

$$\sum_{p\le x}\frac{\log p}{p^s-1},\qquad \prod_{p\le x}(1-p^{-s}).$$

First, let us assume all the known results about the primes, viz.

$$\begin{cases}\vartheta(x) & = x+O\left\{xe^{-a\sqrt{\log x}}\right\}\\ \pi(x) & = \mathrm{Li}(x)\,x+O\left\{xe^{-a\sqrt{\log x}}\right\}.\end{cases}\tag{11.5.1}$$

We have from (11.4.2)

$$\sum_{p\le x}\frac{\log p}{p^s-1}=\frac{\vartheta(x)}{x^s-1}+C(s)-\int\vartheta(x)d\left(\frac{1}{x^s-1}\right),$$

where $C(s)$ is a constant depending on s only. In other words

$$\begin{aligned}\sum_{p\le x}\frac{\log p}{p^s-1}&=C(s)+\frac{x}{x^s-1}-\int xd\left(\frac{1}{x^s-1}\right)\\ &\quad+\frac{O(xe^{-a\sqrt{\log x}})}{x^s-1}+\int O(xe^{-a\sqrt{\log x}})d\left(\frac{1}{x^s-1}\right)\\ &=C(s)+\int\frac{dx}{x^s-1}+O(x^{1-s}e^{-a\sqrt{\log x}})\\ &=C(s)+\frac{x^{1-s}-1}{1-s}+O(x^{1-s}e^{-a\sqrt{\log x}}).\end{aligned}\tag{11.5.2}$$

If $s>1$, then $C(s)$ is obviously

$$\frac{1}{1-s}-\frac{\zeta'(s)}{\zeta(s)},$$

and so

$$\sum_{p\le x}\frac{\log p}{p^s-1}=-\frac{\zeta'(s)}{\zeta(s)}+\frac{x^{1-s}}{1-s}+O(x^{1-s}e^{-a\sqrt{\log x}}).\tag{11.5.3}$$

If $s>1$, then

$$\sum_{p\le x}\frac{\log p}{p^s-1}=\frac{x^{1-s}}{1-s}+O(x^{1-s}e^{-a\sqrt{\log x}}).\tag{11.5.4}$$

If $s=1$, it is easy to see from (11.5.2) and (11.5.3) that

$$\sum_{p\le x}\frac{\log p}{p^s-1}=\log x-\gamma+O(e^{-a\sqrt{\log x}}).\tag{11.5.5}$$

Similarly from (11.4.1) we can show that, if $s < 1$, then

$$\prod_{p \le x} \frac{1}{1 - p^{-s}} = e^{\mathrm{Li}(x^{1-s})} + O(x^{1-s} e^{-a\sqrt{\log x}}), \tag{11.5.6}$$

if $s > 1$, then

$$\prod_{p \le x} \frac{1}{1 - p^{-s}} = \zeta(s) \left\{ 1 + \mathrm{Li}(x^{1-s}) + O(x^{1-s} e^{-a\sqrt{\log x}}) \right\}, \tag{11.5.7}$$

and if $s = 1$, then

$$\prod_{p \le x} \frac{1}{1 - p^{-s}} = e^{\gamma} \log x + O(e^{-a\sqrt{\log x}}), \tag{11.5.8}$$

using

$$\lim_{\epsilon \to 0} \{\mathrm{Li}(1 \pm \epsilon) - \log |\epsilon|\} = \gamma. \tag{11.5.9}$$

11.6 Commentary on the First Paper

The function ϕ is a generic function; but in Sect. 11.2, $\phi(x) = \log x$.

The identity (11.2.2) follows from partial summation. More precisely, by an integration by parts,

$$\vartheta(x) = \int_{2^-}^{x} \log t \, d\pi(t) = \log x \, \pi(x) - \int_{2^-}^{x} \frac{\pi(t)}{t} dt.$$

To prove (11.2.3), we can use Chebyshev's theorem: There exist constants $a, b > 0$ such that for x sufficiently large,

$$\frac{ax}{\log x} < \pi(x) = n < \frac{bx}{\log x}.$$

Taking logarithms throughout, we find that

$$\log a + \log x - \log\log x < \log n < \log b + \log x - \log\log x,$$

from which it follows that

$$\log n = \log x - \log\log x + O(1). \tag{11.6.1}$$

Taking the logarithm of both sides above, we find that

$$\begin{aligned} \log\log n &= \log\log x + \log\left(1 - \frac{\log\log x + O(1)}{\log x}\right) \\ &= \log\log x + O(1). \end{aligned} \tag{11.6.2}$$

Putting (11.6.2) in (11.6.1), we deduce (11.2.3).

The asymptotic formula (11.2.5) does not follow from Ramanujan's previous discourse. In particular,

$$\vartheta(x) \sim n \log n \sim \pi(x) \log x \sim x$$

if and only if $\pi(x) \sim x/\log x$. In other words, (11.2.5) is equivalent to the prime number theorem. Thus, the insertion of (11.2.5) into his discussion is perhaps slightly misleading.

In the definition of $\mathrm{Li}(x)$, it is assumed that the principal value of the integral is taken.

Remember that Ramanujan has assumed the Riemann Hypothesis in recording (11.3.4). There are misprints in (11.3.5), because the error terms are of the same order of magnitude as the main terms. The correct error terms should be

$$O\left(\frac{\sqrt{x}}{\log^3 x}\right).$$

Furthermore, Ramanujan used the Riemann Hypothesis in rearranging the sum on ρ in (11.3.6). The evaluation of that sum follows readily from calculations that can be found in H. Davenport's text [101, pp. 80–81]. The first equality sign in (11.3.7) should be replaced by the inequality sign $\leq$, while the second equality sign in (11.3.7) should be replaced by $\geq$.

At the beginning of Sect. 11.4, Ramanujan more properly should have written $\phi'(t)$ instead of $\phi'(x)$. This kind of inconsistency is common in Ramanujan's writing, both in his notebooks and in his published papers. Usually, no confusion arises from such anomalous writing.

The constant term $C(s)$ in (11.5.2) is equal to 0 if the lower limit is taken to be 2^- in the representation of the far left side as a Riemann–Stieltjes integral. More precisely, by an integration by parts,

$$\sum_{p\leq x} \frac{\log p}{p^s - 1} = \int_{2^-}^{x} \frac{d\theta(t)}{t^s - 1} = \frac{\theta(x)}{x^s - 1} - \int_{2^-}^{x} \theta(t) \frac{d}{dt}\left(\frac{1}{t^s - 1}\right). \tag{11.6.3}$$

As indicated in (11.5.2), Ramanujan then uses the prime number theorem in the form employing $\theta(x)$. Ramanujan did not specify any limits in the integrals, and so if he had chosen a lower limit greater than 2, then $C(s)$ would not be identically equal to 0. But at any rate, no matter what limits are chosen, his following claim, that for $s > 1$,

$$C(s) = \frac{1}{1-s} - \frac{\zeta'(s)}{\zeta(s)},$$

is not justified.

The claim (11.5.4) is false, as can be seen, for example, by letting x tend to ∞. Also note that Ramanujan provides only one term in the Laurent expansion of $(x^{1-s} - 1)/(1 - s)$ about $s = 1$, vitiating his claim of validity.

Upon letting $s \to 1$ in (11.5.6), we see that (11.5.6) is incorrect. In particular, it conflicts with (11.5.8), which is originally due to F. Mertens.

The assertion (11.5.9) follows from a well-known integral representation for Euler's constant [119, p. 31], namely

$$\int_0^1 \left(\frac{1}{\log x} + \frac{1}{1-x} \right) dx = \gamma.$$

11.7 Section 1, Second Paper

[$\phi'(x)$ is a monotonic and continuous function such that $\log |\phi'(x)| = O(x)$.]

Consider the function

$$\begin{aligned} F(x) = \sum_{p \le x} \phi\{\vartheta(p)\} \log p - \frac{1}{2}\phi\{\vartheta(x)\} \log x - \int^{\vartheta(x)} \phi(z) dz \\ + i \log x \int_0^\infty \frac{\phi(\vartheta(x) + iz \log x) - \phi(\vartheta(x) - iz \log x)}{e^{2\pi z} - 1} dz. \end{aligned}$$

Now $F(x)$ is obviously a continuous function except when x is of the form p. Also

$$\lim_{\epsilon \to 0} F(p + \epsilon) = F(p).$$

Again

$$\begin{aligned} \lim_{\epsilon \to 0} F(p - \epsilon) &= F(p) - \phi\{\vartheta(p)\} \log p - \frac{1}{2}\phi\{\vartheta(p) - \log p\} \log p \\ &+ \frac{1}{2}\phi\{\vartheta(p)\} \log p + \int_{\vartheta(p) - \log p}^{\vartheta(p)} \phi(z) dz \\ &+ i \log p \int_0^\infty \frac{\phi(\vartheta(p) - \log p + iz \log p) - \phi(\vartheta(p) - \log p - iz \log p)}{e^{2\pi z} - 1} dz \\ &- i \log p \int_0^\infty \frac{\phi(\vartheta(p) + iz \log p) - \phi(\vartheta(p) - iz \log p)}{e^{2\pi z} - 1} dz. \end{aligned}$$

But it is well known that

$$\begin{aligned} \frac{1}{h} \int_{x-h}^x \phi(z) dz = \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x - h) + i \int_0^\infty \frac{\phi(x + ihz) - \phi(x - ihz)}{e^{2\pi z} - 1} dz \\ - i \int_0^\infty \frac{\phi(x - h + ihz) - \phi(x - h - ihz)}{e^{2\pi z} - 1} dz. \end{aligned} \tag{11.7.1}$$

Hence

$$\lim_{\epsilon \to 0} F(p - \epsilon) = F(p).$$

Thus we see that $F(x)$ is continuous without exception.

Now the derived function $\overline{F}(x)$ viz.

$$-\frac{1}{2x}\phi\{\vartheta(x)\}+\frac{i}{x}\int_0^\infty \frac{\phi\{\vartheta(x)+iz\log x\}-\phi\{\vartheta(x)-iz\log x\}}{e^{2\pi z}-1}dz \\ -\frac{\log x}{x}\int_0^\infty z\frac{\phi'\{\vartheta(x)+iz\log x\}+\phi'\{\vartheta(x)-iz\log x\}}{e^{2\pi z}-1}dz$$

is finite and continuous except at the isolated points $x=p$. Hence

$$F(x)=\int^x \overline{F}(x)dx.$$

That is to say

$$\sum_{p\le x}\phi\{\vartheta(p)\}\log p=\int_0^{\vartheta(x)}\phi(z)dz+\frac{1}{2}\phi\{\vartheta(x)\}\log x \\ -\frac{1}{2}\int_1^x \phi\{\vartheta(z)\}\frac{dz}{z}+R(x) \qquad (11.7.2)$$

where

$$R(x)=\log x\int_0^\infty \frac{\phi\{\vartheta(x)+iz\log x\}-\phi\{\vartheta(x)-iz\log x\}}{i(e^{2\pi z}-1)}dz \\ -\int_1^x \frac{dy}{y}\int_0^\infty \frac{\phi\{\vartheta(y)+iz\log y\}-\phi\{\vartheta(y)-iz\log y\}}{i(e^{2\pi z}-1)}dz \\ -\int_1^x \frac{\log y}{y}dy\int_0^\infty z\frac{\phi'\{\vartheta(y)+iz\log y\}+\phi'\{\vartheta(y)-iz\log y\}}{e^{2\pi z}-1}dz.$$

Since the two sides vanish when $x=1$, no constant is required. But if the integrals are either divergent or meaningless near the beginning, then we have to adjust the constant after choosing suitable lower limits for the integrals.

11.8 Section 2, Second Paper

It is easy to see that

$$R(x)=O\left[|\phi'\{\vartheta(x)\}|(\log x)^2\right]. \qquad (11.8.1)$$

From this and (11.7.2) it follows that

$$\sum_{p\le x}\phi\{\vartheta(p)\}\log p=C+\int^{\vartheta(x)+\frac{1}{2}\log x}\phi(z)dz-\frac{1}{2}\int^x \frac{1}{z}\phi\{\vartheta(z)\}dz \\ +O\left[|\phi'\{\vartheta(x)\}|(\log x)^2\right]. \qquad (11.8.2)$$

For example when $\phi(x) = x^s$, we have

$$\sum_{p\le x}\{\vartheta(p)\}^s \log p = C + \frac{1}{s+1}\{\vartheta^{s+1}(x) - 1\} + \frac{1}{2}\vartheta^s(x)\log x$$
$$- \frac{1}{2}\int^x \frac{1}{z}\vartheta^s(z)dz + O\{x^{s-1}(\log x)^2\}$$
$$= C + \frac{1}{s+1}\{\vartheta^{s+1}(x) - 1\} + \frac{1}{2}\vartheta^s(x)\log x + O(x^s) \quad (11.8.3)$$

for all values of s except 0. Here we have not assumed the prime number theorem. If we assume the Riemann Hypothesis we can easily show that the result is

$$C + \frac{1}{s+1}\{\vartheta^{s+1}(x) - 1\} + \frac{1}{2}\vartheta^s(x)\log x - \frac{x^s - 1}{2s} + O(x^{s-1/2}) \quad (11.8.4)$$

for all values of s. (C depends on s only.)

11.9 Section 3, Second Paper

Another very interesting case is when $\phi = e^{-sx}$. From (11.7.2) we have

$$\sum_{p\le x} e^{-s\vartheta(p)}\log p = \int_0^{\vartheta(x)} e^{-sz}dz + \frac{1}{2}e^{-s\vartheta(x)}\log x$$
$$- \frac{1}{2}\int_1^x e^{-s\vartheta(z)}\frac{dz}{z} + R(x) \quad (11.9.1)$$

where

$$R(x) = -2\log x\, e^{-s\vartheta(x)}\int_0^\infty \frac{\sin(zs\log x)}{e^{2\pi z} - 1}dz$$
$$+ 2\int_1^x \frac{1}{y}e^{-s\vartheta(y)}dy\int_0^\infty \frac{\sin(zs\log y)}{e^{2\pi z} - 1}dz$$
$$+ 2s\int_1^x \frac{1}{y}\log y e^{-s\vartheta(y)}dy\int_0^\infty \frac{z\cos(zs\log y)}{e^{2\pi z} - 1}dz. \quad (11.9.2)$$

It follows that

$$\sum_{p\le x} e^{-s\vartheta(p)}\log p = \frac{1}{s} - \frac{\log x}{x^s - 1}e^{-s\vartheta(x)} - \int_1^x \frac{1 - z^s + sz^s\log z}{z(z^s - 1)^2}e^{-s\vartheta(z)}dz. \quad (11.9.3)$$

This suggests a more general case viz.

$$\sum_{p\le x} e^{-s\vartheta(p)} f(p) = \frac{f(2)}{2^s-1} + e^{-s\vartheta(2)}\left\{\frac{f(3)}{3^s-1} - \frac{f(2)}{2^s-1}\right\}$$
$$+ e^{-s\vartheta(3)}\left\{\frac{f(5)}{5^s-1} - \frac{f(3)}{3^s-1}\right\} + e^{-s\vartheta(5)}\left\{\frac{f(7)}{7^s-1} - \frac{f(5)}{5^s-1}\right\}$$
$$+ \cdots + e^{-s\vartheta(p')}\left\{\frac{f(p)}{p^s-1} - \frac{f(p')}{p'^s-1}\right\}$$
$$+ e^{-s\vartheta(p)}\left\{\frac{f(x)}{x^s-1} - \frac{f(p)}{p^s-1}\right\} - \frac{f(x)}{x^s-1}e^{-s\vartheta(x)}$$

where p' is the prime next below p. In other words

$$\begin{aligned}\sum_{p\le x} e^{-s\vartheta(p)} f(p) &= \frac{f(2)}{2^s-1} - \frac{f(x)}{x^s-1}e^{-s\vartheta(x)} + \int_2^x e^{-s\vartheta(z)}\left(\frac{d}{dz}\frac{f(z)}{z^s-1}\right)dz \\ &= \frac{f(2)}{2^s-1} - \frac{f(x)}{x^s-1}e^{-s\vartheta(x)} + \int_2^x e^{-s\vartheta(z)}\left\{\frac{f'(z)}{z^s-1} - \frac{sz^{s-1}f(z)}{(z^s-1)^2}\right\}dz.\end{aligned} \tag{11.9.4}$$

If we suppose that $f(x)=1$ in (11.9.4), then

$$\sum_{p\le x} e^{-s\vartheta(p)} = \frac{1}{2^s-1} - \frac{e^{-s\vartheta(x)}}{x^s-1} - s\int_2^x \frac{z^{s-1}e^{-s\vartheta(z)}}{(z^s-1)^2}dz. \tag{11.9.5}$$

11.10 Section 4, Second Paper

Making x tend to ∞ in (11.7.2) and (11.9.4), we obtain

$$\begin{aligned}&\phi\{\vartheta(2)\}\log 2 + \phi\{\vartheta(3)\}\log 3 + \phi\{\vartheta(5)\}\log 5 + \phi\{\vartheta(7)\}\log 7 + \cdots \\ &= \int_0^\infty \phi(x)dx - \frac{1}{2}\int_1^\infty \frac{1}{x}\phi\left(\vartheta(x)\right)dx \\ &\quad - \int_1^\infty \frac{dx}{x}\int_0^\infty \frac{\phi(\vartheta(x)+iz\log x) - \phi(\vartheta(x)-iz\log x)}{i(e^{2\pi z}-1)}dz \\ &\quad - \int_1^\infty \frac{\log x}{x}dx\int_0^\infty z\frac{\phi'(\vartheta(x)+iz\log x) + \phi'(\vartheta(x)-iz\log x)}{e^{2\pi z}-1}dz,\end{aligned} \tag{11.10.1}$$

$$\begin{aligned}&2^{-s}f(2) + 6^{-s}f(3) + 30^{-s}f(5) + 210^{-s}f(7) + \cdots \\ &= \frac{f(2)}{2^s-1} + \int_2^\infty e^{-s\vartheta(x)}\left\{\frac{f'(x)}{x^s-1} - \frac{sx^{s-1}f(x)}{(x^s-1)^2}\right\}dx.\end{aligned} \tag{11.10.2}$$

As particular cases of (11.10.2) we have

$$2^{-s} + 6^{-s} + 30^{-s} + 210^{-s} + \cdots = \frac{1}{2^s - 1} - s\int_2^\infty \frac{x^{s-1}e^{-s\vartheta(x)}}{(x^s-1)^2}dx, \quad (11.10.3)$$

$$\begin{aligned} &2^{-s}\log 2 + 6^{-s}\log 3 + 30^{-s}\log 5 + 210^{-s}\log 7 + \cdots \\ &= \frac{1}{s} - \int_1^\infty e^{-s\vartheta(x)}\frac{1 - x^s + sx^s\log x}{x(1-x^s)^2}dx. \end{aligned} \quad (11.10.4)$$

11.11 Section 5, Second Paper

Let us consider the behaviour of (11.10.4) and (11.10.3) as $s \to 0$.

$$\begin{aligned} &2^{-s}\log 2 + 6^{-s}\log 3 + 30^{-s}\log 5 + \cdots \\ &= \frac{1}{s} - \int_1^\infty e^{-s\vartheta(x)}\left\{\frac{1}{2} - \frac{s\log x}{6} + \frac{s^3(\log x)^3}{180} - \cdots\right\}\frac{dx}{x} \\ &= \frac{1}{s} - \frac{1}{2}\int_1^\infty e^{-s\vartheta(x)}\frac{dx}{x} + O\{s(\log s)^2\} \\ &= \frac{1}{s} + \frac{1}{2}\log s + C + O(\sqrt{s}) \qquad \text{(Riemann's Hypothesis)} \\ &= \frac{1}{s} + \frac{1}{2}\log s + C + O\{e^{-a\sqrt{\log(1/s)}}\} \qquad \text{(Prime number theorem)} \\ &= \frac{1}{s} + \frac{1}{2}\log s + O(1) \qquad \text{(Elementary reasoning).} \end{aligned} \quad (11.11.1)$$

Again,

$$\begin{aligned} &2^{-s} + 6^{-s} + 30^{-s} + 210^{-s} + \cdots \\ &= \frac{1}{2^s-1} - \int_2^\infty e^{-s\vartheta(x)}\left\{\frac{1}{s(\log x)^2} - \frac{s}{12} + \frac{s^3(\log x)^2}{240} - \cdots\right\}\frac{dx}{x} \\ &= \frac{1}{2^s-1} - \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{x(\log x)^2}dx + O(s\log s) \\ &= \frac{1}{2^s-1} - \frac{1}{s}\int_2^\infty \frac{e^{-sx}}{x(\log x)^2}dx + O\left(\frac{e^{-a\sqrt{\log(1/s)}}}{s}\right) \\ &= \frac{1}{2^s-1} - \frac{e^{-2s}}{s\log 2} + \int_2^\infty \frac{e^{-sx}}{\log x}dx + O\left\{\frac{e^{-a\sqrt{\log(1/s)}}}{s}\right\} \\ &= -\frac{1}{s\log s} + \frac{\gamma_1}{s(\log s)^2} - \frac{\gamma_2}{s(\log s)^3} + \frac{\gamma_3}{s(\log s)^4} - \cdots + O\left\{\frac{1}{s(\log(1/s))^k}\right\}; \end{aligned} \quad (11.11.2)$$

where

$$\Gamma(1+z) = 1 - \frac{\gamma_1 z}{1!} + \frac{\gamma_2 z^2}{2!} - \frac{\gamma_3 z^3}{3!} + \cdots,$$

so that γ_1 is Euler's constant. Without assuming the prime number theorem we can prove that (11.11.2) is of the form

$$-\frac{1}{s\log s} + O\left\{\frac{1}{s(\log s)^2}\right\}. \tag{11.11.3}$$

Hence

$$(1-2^{-s})(1-6^{-s})(1-30^{-s})(1-210^{-s})\cdots = \exp\left(\frac{\pi^2}{6s\log s}\right) + O\left\{\frac{1}{s(\log s)^2}\right\}. \tag{11.11.4}$$

11.12 Commentary on the Second Paper

Ramanujan tacitly assumes that $\epsilon > 0$ throughout his paper.

The "well known" identity (11.7.1) is a special case of Plana's summation formula [315, p. 145, Exercise 7].

The Eq. (11.8.2) is inexplicably wrong. Evidently, Ramanujan had intended to write another version of (11.7.2) in terms of indefinite integrals, and so why he wrote (11.8.2) is a mystery. Fortunately, it is not used in the sequel.

We briefly sketch a proof of (11.9.3). We use the evaluations [126, p. 516, formula 3.911, no. 2; p. 527, formula 3.951, no. 5]

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}-1}dx = \frac{\pi}{2b}\coth\left(\frac{\pi a}{b}\right) - \frac{1}{2a}, \qquad a,b>0,$$
$$\int_0^\infty \frac{x\cos(ax)}{e^{bx}-1}dx = \frac{1}{2a^2} - \frac{\pi^2}{2b^2}\operatorname{csch}^2\left(\frac{\pi a}{b}\right), \qquad b>0.$$

Thus,

$$\begin{aligned}\int_0^\infty \frac{\sin(zs\log x)}{e^{2\pi z}-1}dz &= \frac{1}{4}\coth\left(\frac{s\log x}{2}\right) - \frac{1}{2s\log x}\\ &= \frac{1}{4}\frac{x^s+1}{x^s-1} - \frac{1}{2s\log x}\end{aligned} \tag{11.12.1}$$

and

$$\begin{aligned}\int_0^\infty \frac{z\cos(zs\log y)}{e^{2\pi z}-1}dz &= \frac{1}{2s^2\log^2 y} - \frac{1}{8}\operatorname{csch}^2\left(\frac{s\log y}{2}\right)\\ &= \frac{1}{2s^2\log^2 y} - \frac{y^s}{2(y^s-1)^2}.\end{aligned} \tag{11.12.2}$$

Hence, by (11.12.1) and (11.12.2),

$$
\begin{aligned}
&-2\log x\int_0^\infty \frac{\sin(zs\log x)}{e^{2\pi z}-1}dz=\frac{1}{s}-\frac{\log x}{2}\frac{x^s+1}{x^s-1},\\
&2\int_1^x \frac{1}{y}e^{-s\vartheta(y)}dy\int_0^\infty \frac{\sin(zs\log x)}{e^{2\pi z}-1}dz\\
&\quad=\int_1^x e^{-s\vartheta(y)}\left(-\frac{1}{ys\log y}+\frac{1}{2y}\frac{y^s+1}{y^s-1}\right)dy,\\
&2s\int_1^x \frac{\log y}{y}e^{-s\vartheta(y)}dy\int_0^\infty \frac{z\cos(zs\log y)}{e^{2\pi z}-1}dz\\
&\quad=\int_1^x e^{-s\vartheta(y)}\left(\frac{1}{ys\log y}-\frac{sy^s\log y}{(y^s-1)^2}\right)dy.
\end{aligned}
$$

Using the three calculations above in (11.9.2), then putting the resulting formula for $R(x)$ in (11.9.1), evaluating the elementary integral on the right-hand side of (11.9.1), and considerably simplifying, we arrive at (11.9.3).

We explain how (11.10.4) was obtained. Note that Ramanujan added and subtracted a certain integral with limits 1 and 2. Using (11.9.4), we observe that

$$
\begin{aligned}
\int_1^2 \frac{1-x^s+sx^s\log x}{x(1-x^s)^2}dx&=-\int_1^2\frac{d}{dx}\left(\frac{\log x}{x^s-1}\right)dx\\
&=-\frac{\log 2}{2^s-1}+\lim_{x\to1}\frac{\log x}{x^s-1}=-\frac{\log 2}{2^s-1}+\frac{1}{s}.
\end{aligned}
$$

Using the calculation above in (11.10.2) with $f(x)=\log x$, we deduce (11.10.4) at once.

On the right-hand side of the third equality in (11.11.1), on the Riemann Hypothesis, it would seem that the error term should be $O(\log^2 s\sqrt{s})$. For the fourth equality, Ramanujan invoked the classical error term of de la Valleé Poussin [101, p. 113].

Details will now be provided for the last line of (11.11.2). First observe that

$$
\lim_{s\to0}\left(\frac{1}{2^s-1}-\frac{e^{-2s}}{s\log 2}\right)=1.
$$

Second, write

$$
\int_2^\infty \frac{e^{-sx}}{\log x}dx=\int_0^\infty \frac{e^{-sx}}{\log x}dx+O(1),
$$

as $s\to0$, provided the integral on the right-hand side above is interpreted as a principal value. Write

$$\begin{aligned}
\int_0^\infty \frac{e^{-sx}}{\log x} dx &= -\frac{1}{s\log s}\int_0^\infty \frac{e^{-u}}{1-\log u/\log s} du \\
&= -\frac{1}{s\log s}\int_0^\infty e^{-u}\left(\sum_{j=0}^{k-1}\left(\frac{\log u}{\log s}\right)^j + \left(\frac{\log u}{\log s}\right)^k \frac{1}{1-\log u/\log s}\right) du \\
&= -\frac{1}{s\log s}\sum_{j=0}^{k-1}\frac{1}{\log^j s}\int_0^\infty e^{-u}\log^j u\, du + O\left(\frac{1}{s\log^{k+1}(1/s)}\right) \\
&= -\frac{1}{s\log s}\sum_{j=0}^{k-1}\frac{\Gamma^{(j)}(1)}{\log^j s} + O\left(\frac{1}{s\log^{k+1}(1/s)}\right),
\end{aligned}$$

from which Ramanujan's claim in (11.11.2) follows. It is not clear why Ramanujan did not define $\gamma_j = \Gamma^{(j)}(1)$ instead of $\gamma_j = (-1)^j\Gamma^{(j)}(1)$.

We do not think that Ramanujan's claim in (11.11.3) is justified, i.e., some form of the prime number theorem seems necessary to achieve (11.11.3).

We now prove (11.11.4). Using (11.11.3), we find that

$$\begin{aligned}
\log\prod_{j=1}^{\infty}(1-(p_1p_2\cdots p_j)^{-s}) &= \sum_{j=1}^{\infty}\log(1-(p_1p_2\cdots p_j)^{-s}) \\
&= -\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}(p_1p_2\cdots p_j)^{-ns} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n\cdot ns\log(ns)} + O\left(\frac{1}{n\cdot ns\log^2(ns)}\right)\right) \\
&= \frac{\pi^2}{6s\,\log s} + O\left(\frac{1}{s\log^2 s}\right).
\end{aligned}$$

Exponentiating both sides above, we deduce (11.11.4).

12

An Unpublished Manuscript of Ramanujan on Infinite Series Identities

12.1 Introduction

Published with Ramanujan's lost notebook [269, pp. 318–321] is a four-page, handwritten fragment on infinite series. Partial fraction expansions, the Riemann zeta function $\zeta(s)$, alternating sums over the odd integers, divisor sums $\sigma_k(n)$, Bernoulli numbers, and Euler numbers are featured in the formulas in this manuscript. The first result has the equation number (18) attached to it. Thus, the manuscript was likely intended to be the completion of either a published paper or another unpublished manuscript. We conjecture that this fragment was originally intended to be a part of Ramanujan's paper *Some formulae in the analytic theory of numbers*, [263], [267, pp. 133–135]. This paper contains several theorems featuring $\zeta(s)$ and $\sigma_k(n)$, and so the topics in the unpublished manuscript mesh well with those in the published paper. However, the last tagged equation in [263] is (22), whereas we would expect it to be (17) if our conjecture is correct. Often Ramanujan would think of additional results and add them to the paper as he was writing it, and so this could easily account for the discrepancy in equation numbers. We remark here that the manuscript does not provide any proofs, but Ramanujan usually gives an indication (in one line) how a particular formula may be deduced.

Why did Ramanujan not include this discarded piece in his paper [263], for the published paper is rather short, and the unpublished manuscript would add at most four pages to the length of the paper? We think that Ramanujan discovered that one of his claims, namely (21), was incorrect and that two of his deductions were not corollaries of his (incorrect) formula, as he had previously thought. Moreover, we suspect that he realized that some of his arguments were not rigorous. Since he had abandoned his intention to publish this portion, he did not bother to indicate that changes or corrections needed to be made in the fragment. He probably failed to discard it because he had wanted to return to it sometime in the future to attempt to correct his arguments.

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Ramanujan loved partial fraction expansions. Chapter 14 in his second notebook [38, 268], in particular, contains several such expansions, and others are scattered throughout all three earlier notebooks. See [40, Chap. 30] for some of these scattered partial fraction decompositions. However, Ramanujan's arguments were not always rigorous. Because of his apparent weakness in complex analysis, he evidently did not have a firm grasp of the Mittag–Leffler theorem, for claim (21) in his unpublished manuscript arises from an incorrect application of the Mittag–Leffler theorem, as we detail below. After claim (21), he then asserted several corollaries arising from this (incorrect) partial fraction decomposition. All of the corollaries are indeed correct, but two of them do not follow from this partial fraction expansion. Ramanujan undoubtedly had previously been familiar with all of these corollaries and almost certainly had derived them by other methods. Certain correct results were easily deduced from his expansion, and he must have been puzzled why two further known results could not be similarly deduced. It is interesting that the same incorrect partial fraction expansion occurs in Entry 19(i) of Chap. 14 of his second notebook [268], [38, p. 271], where it was derived by a different method, namely a general elementary theorem, Entry 18 of Chap. 14 [268], [38, pp. 267–268]. R. Sitaramachandrarao [289], [38, pp. 271–272] found an alternative version of Ramanujan's partial fraction expansion. After we provide Ramanujan's argument, we show that we can actually use Sitaramachandrarao's result to derive a corrected version of Ramanujan's partial fraction expansion. We shall see that Ramanujan's defective argument missed one expression; all other portions of Ramanujan's formula are correct. One of the two claims that did not follow from Ramanujan's expansion now is a corollary of the corrected version. However, this corrected version still does not allow us to rigorously deduce the other result.

The most celebrated result in this manuscript is probably claim (28), which is a famous formula for $\zeta(2n+1)$, where n is a positive integer. There is a large number of proofs of this result and many generalizations as well. References are given after we provide Ramanujan's proof of (28). Ramanujan's argument is rigorous and ironically is independent of whether his formula or the corrected version is used.

In (22), Ramanujan gives another partial fraction expansion, but this one is correct. All of its corollaries claimed by Ramanujan are correct, but not all the deductions can be rigorously established by Ramanujan's methods. These corollaries, like those arising from (19), are all well known, with some having been proved in the literature several times.

In the remainder of the chapter, we record all of Ramanujan's formulas, prove them rigorously in some cases, and "prove" them nonrigorously in other cases, i.e., we argue as Ramanujan most likely did. Most of the results appear in Ramanujan's notebooks, and for all theorems we provide references where proofs can be found. In providing references, we have adhered to the following rules. For each principal theorem, we locate it in Ramanujan's notebooks, indicate who gave the first proof, and lastly refer to the pages in the second author's books, primarily [38], where references to further proofs can be found. Since the publication of [38], additional proofs have been found in some instances, and so we provide references to those recent proofs of which we are aware.

The residue of a meromorphic function $f(z)$ at a pole z_0 will be denoted by $R(f, z_0) = R(z_0)$.

12.2 Three Formulas Containing Divisor Sums

Entry 12.2.1 (p. 318, formula (18)). *Let $\chi(n)$ denote the nonprincipal primitive character of modulus 4, i.e., $\chi(2n) = 0$ and $\chi(2n+1) = (-1)^n$, for each nonnegative integer n. Let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if $x \neq in$, for each integer n,*

$$\sum_{n=1}^{\infty} \frac{\chi(n)d(n)n}{n^2+x^2} = \frac{\pi}{4}\sum_{n=1}^{\infty} \frac{\chi(n)}{n}\operatorname{sech}\left(\frac{\pi x}{2n}\right). \tag{12.2.1}$$

Proof. Recall the partial fraction expansion [126, p. 44, formula 1.422, no. 1]

$$\operatorname{sech}\left(\frac{\pi x}{2}\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2k-1}{(2k-1)^2+x^2}.$$

Thus,

$$\begin{aligned}\frac{\pi}{4}\sum_{n=1}^{\infty} \frac{\chi(n)}{n}\operatorname{sech}\left(\frac{\pi x}{2n}\right) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sum_{k=1}^{\infty}\frac{\chi(k)k}{k^2+x^2/n^2}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(nk)nk}{n^2k^2+x^2}\\ &= \sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}.\end{aligned}$$

This formally completes our argument. However, observe that in the penultimate line we rearranged the order of summation in the double sum, and this needs to be justified. The following argument was kindly supplied by Johann Thiel.

Proposition 12.2.1. *Let $\chi(n)$ denote the nonprincipal primitive character of modulus 4. Let $d(n)$ denote the number of divisors of the positive integer n. Then if $x \neq \imath n$, for each integer n,*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(nk)nk}{n^2k^2+x^2}=\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}. \tag{12.2.2}$$

Proof. By the identity theorem, it suffices to show that (12.2.2) holds for $x \in [0, \frac{1}{4}]$.

We first examine the right-hand side of (12.2.2). If N is a positive integer, write

$$\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=\sum_{r=1}^{4N^2}\frac{\chi(r)d(r)r}{r^2+x^2}+\sum_{r=4N^2+1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}. \tag{12.2.3}$$

We want to show that as $N \to \infty$,

$$\sum_{r=4N^2+1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=O\left(\frac{1}{N}\right). \tag{12.2.4}$$

To achieve this, we use the Dirichlet hyperbola method. Write

$$\begin{aligned}
\sum_{n\le y}\chi(n)d(n) &= \sum_{n\le y}\chi(n)\sum_{d|n}1=\sum_{d\le y}\sum_{\substack{n\le y\\ d|n}}\chi(n)\\
&=\sum_{d\le y}\sum_{m\le y/d}\chi(md)=\sum_{\substack{a,b\le y\\ ab\le y}}\chi(ab)\\
&=\sum_{a\le\sqrt{y}}\sum_{b\le y/a}\chi(a)\chi(b)+\sum_{b\le\sqrt{y}}\sum_{a\le y/b}\chi(a)\chi(b)-\sum_{a\le\sqrt{y}}\sum_{b\le\sqrt{y}}\chi(a)\chi(b)\\
&=2\sum_{a\le\sqrt{y}}\chi(a)\sum_{b\le y/a}\chi(b)-\sum_{a\le\sqrt{y}}\sum_{b\le\sqrt{y}}\chi(a)\chi(b)\\
&=O(\sqrt{y}),
\end{aligned} \tag{12.2.5}$$

as $y \to \infty$, where we used the fact that each of the inner sums in the penultimate line is $O(1)$. If we now apply partial summation in a straightforward fashion with the use of (12.2.5), we easily deduce (12.2.4). Using then (12.2.4) back in (12.2.3), we conclude that

$$\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=\sum_{r=1}^{4N^2}\frac{\chi(r)d(r)r}{r^2+x^2}+O\left(\frac{1}{N}\right). \tag{12.2.6}$$

Next, we examine the first sum on the right-hand side of (12.2.3), or the sum on the right-hand side in (12.2.6). Hence,

$$
\begin{aligned}
\sum_{r=1}^{4N^2} \frac{\chi(r)d(r)r}{r^2+x^2} &= \sum_{nk\le 4N^2} \frac{\chi(nk)nk}{n^2k^2+x^2} \\
&= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + 2\sum_{n=1}^{2N-1} \sum_{k=2N+1}^{\left\lfloor \frac{4N^2}{n} \right\rfloor} \frac{\chi(nk)nk}{n^2k^2+x^2} \\
&= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + 2\sum_{n=1}^{2N-1} \frac{\chi(n)}{n} \sum_{k=2N+1}^{\left\lfloor \frac{4N^2}{n} \right\rfloor} \frac{\chi(k)k}{k^2+(x/n)^2}.
\end{aligned}
\tag{12.2.7}
$$

Observe that the inner sum in the second series on the far right side of (12.2.7) is an alternating series and is consequently $O(1/N)$, as $N \to \infty$. Using this bound in (12.2.7) and then (12.2.7) in (12.2.6) gives

$$
\sum_{r=1}^{\infty} \frac{\chi(r)d(r)r}{r^2+x^2} = \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{\log N}{N}\right). \tag{12.2.8}
$$

We now examine the left-hand side of (12.2.2) and readily find that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} &= \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=2N+1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} \\
&= \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=2N+1}^{\infty} \frac{\chi(n)}{n} \sum_{k=1}^{\infty} \frac{\chi(k)k}{k^2+(x/n)^2}.
\end{aligned}
\tag{12.2.9}
$$

If we set

$$
f(y) := \frac{1}{y} \sum_{k=1}^{\infty} \frac{\chi(k)k}{k^2+(x/y)^2},
$$

for $y \in [1,\infty)$, by a straightforward calculation we see that $f'(y) < 0$ and consequently $\lim_{y\to\infty} f(y) = 0$. Therefore, we can apply the alternating series test to conclude that the inner sum of the second sum on the far right side of (12.2.9) is an alternating series that is $O(1/N)$, as $N \to \infty$. Therefore,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} &= \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{1}{N}\right) \\
&= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=1}^{2N}\sum_{k=2N+1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{1}{N}\right) \\
&= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{\log N}{N}\right),
\end{aligned}
\tag{12.2.10}
$$

where the last equality follows by an argument similar to the one used to deduce (12.2.8).

Taking the difference of (12.2.10) and (12.2.8), we complete the proof of (12.2.2). □

This then completes a rigorous proof of Entry 12.2.1. □

Entry 12.2.1 is a simple example of a large class of formulas involving the sech function and arithmetic functions. See papers by Berndt [34, Example 3] and P.V. Krishnaiah and R. Sita Rama Chandra Rao [201] for further examples.

Entry 12.2.2 (p. 318, formula (19)). *Let* $\sigma_k(n) = \sum_{d|n} d^k$. *Then, for* $\operatorname{Re} s > 1$ *and* $\operatorname{Re}(s-r) > 1$,

$$\zeta(s)\zeta(s-r) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}. \tag{12.2.11}$$

The formula (12.2.11) is classical and simple to prove. Ramanujan [263], [267, pp. 133–135] found beautiful extensions of it. See also Titchmarsh's text [306, p. 8].

Entry 12.2.3 (p. 318, formula (20)). *Let* χ *be defined as in Entry* 12.2.1, *and let* $\sigma_k(n)$ *be as in Entry* 12.2.2. *Then, for* $\operatorname{Re} s > 1$ *and* $\operatorname{Re}(s-r) > 1$,

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_r(n)}{n^s}.$$

Proof. For $\operatorname{Re} s > 1$ and $\operatorname{Re}(s-r) > 1$,

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{m,n=1}^{\infty} \frac{\chi(mn)n^r}{(mn)^s} = \sum_{k=1}^{\infty} \frac{\chi(k)\sigma_r(k)}{k^s},$$

which completes the proof for $\operatorname{Re} s > 1$ and $\operatorname{Re}(s-r) > 1$. We expect that the domain of validity can be extended to $\operatorname{Re} s > \sup\{0, \operatorname{Re} r\}$, but we are unable to prove this. □

There are many results in the literature generalizing or extending the last two results. The two most extensive papers in this direction are perhaps those by S. Chowla [91, 92], [95, pp. 92–115, 120–130].

12.3 Ramanujan's Incorrect Partial Fraction Expansion and Ramanujan's Celebrated Formula for $\zeta(2n+1)$

Prior to this next claim, Ramanujan writes, "By the theory of residues it can be shown that". Evidently, Ramanujan implied that he used the residue theorem to calculate the partial fraction decomposition that followed. His formal calculations should depend upon an application of the Mittag–Leffler theorem, which cannot be applied in this situation. We first state the incorrect expansion, indicate Ramanujan's probable approach, and then offer a correct version. Ramanujan used n to denote a complex variable; we replace it with the more natural notation $w = z^2$.

Entry 12.3.1 (p. 318, formula (21)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\frac{1}{2w} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\} = \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}). \tag{12.3.1}$$

Proof. (We emphasize that the following argument is not rigorous.) Consider

$$f(z) := \frac{\pi}{2} \cot(z\sqrt{\alpha}) \coth(z\sqrt{\beta}),$$

which has simple poles at $z = m\pi/\sqrt{\alpha}$, $-\infty < m < \infty$, $m \neq 0$, with residues

$$R(m\pi/\sqrt{\alpha}) = \frac{\pi}{2\sqrt{\alpha}} \coth(m\beta), \tag{12.3.2}$$

and simple poles at $z = m\pi i/\sqrt{\beta}, -\infty < m < \infty, m \neq 0$, with residues

$$R(m\pi i/\sqrt{\beta}) = -\frac{\pi i}{2\sqrt{\beta}} \coth(m\alpha), \tag{12.3.3}$$

where we used the fact $\alpha\beta = \pi^2$ in our calculations. Clearly $f(z)$ also has a double pole at $z = 0$. Using (12.3.2) and once again the relation $\alpha\beta = \pi^2$, we find that the contributions of the poles $z = m\pi/\sqrt{\alpha}$ and $z = -m\pi/\sqrt{\alpha}$, $1 \leq m < \infty$, to the partial fraction expansion of $f(z)$ are

$$\frac{\pi}{2\sqrt{\alpha}} \left(\frac{\coth(m\beta)}{z - m\pi/\sqrt{\alpha}} + \frac{\coth(-m\beta)}{z + m\pi/\sqrt{\alpha}} \right) = \frac{m\beta \coth(m\beta)}{z^2 - m^2\beta}. \tag{12.3.4}$$

Using (12.3.3) and once again the relation $\alpha\beta = \pi^2$, we find that the sum of the contributions of the poles $z = m\pi i/\sqrt{\beta}$ and $z = -m\pi i/\sqrt{\beta}$, $1 \leq m < \infty$, to the partial fraction decomposition of $f(z)$ equals

$$-\frac{\pi i}{2\sqrt{\beta}} \left(\frac{\coth(m\alpha)}{z - m\pi i/\sqrt{\beta}} - \frac{\coth(m\alpha)}{z + m\pi i/\sqrt{\beta}} \right) = \frac{m\alpha \coth(m\alpha)}{z^2 + m^2\alpha}. \tag{12.3.5}$$

That part of the partial fraction decomposition arising from the double pole at $z = 0$ clearly equals

$$\frac{\pi}{2\sqrt{\alpha\beta}z^2} = \frac{1}{2z^2}, \tag{12.3.6}$$

upon again using the relation $\alpha\beta = \pi^2$. Employing (12.3.4)–(12.3.6) and applying the Mittag–Leffler theorem, we find that there exists an entire function $g(z)$ such that

$$\frac{\pi}{2}\cot(z\sqrt{\alpha})\coth(z\sqrt{\beta}) = \frac{1}{2z^2} + \sum_{m=1}^{\infty}\left\{\frac{m\alpha\coth(m\alpha)}{z^2+m^2\alpha} + \frac{m\beta\coth(m\beta)}{z^2-m^2\beta}\right\} + g(z). \tag{12.3.7}$$

Here Ramanujan probably assumed that $g(z) \equiv 0$ and so completed his "proof" of (12.3.1). □

Normally, in applications of the Mittag–Leffler theorem, one lets $z \to \infty$ to conclude that $g(z) \equiv 0$. However, such an argument is invalid here, because $\cot(z\sqrt{\alpha})\coth(z\sqrt{\beta})$ oscillates and does not have a limit as $z \to \infty$. Moreover, one cannot justify taking the limit as $z \to \infty$ under the summation sign in (12.3.7).

In attempting to find a corrected version of (12.3.1), Sitaramachandrarao [289], [38, pp. 271–272] proved that

$$\pi^2 xy\cot(\pi x)\coth(\pi y) = 1 + \frac{\pi^2}{3}(y^2 - x^2) - 2\pi xy\sum_{m=1}^{\infty}\left(\frac{y^2\coth(\pi m x/y)}{m(m^2+y^2)} + \frac{x^2\coth(\pi m y/x)}{m(m^2-x^2)}\right). \tag{12.3.8}$$

Using the elementary identities

$$\frac{y^2}{m(m^2+y^2)} = -\frac{m}{m^2+y^2} + \frac{1}{m}$$

and

$$\frac{x^2}{m(m^2-x^2)} = \frac{m}{m^2-x^2} - \frac{1}{m},$$

we find that (12.3.8) can be rewritten in the form

$$\begin{aligned}\pi^2 xy\cot(\pi x)\coth(\pi y) &= 1 + \frac{\pi^2}{3}(y^2 - x^2) \\ &\quad + 2\pi xy\sum_{m=1}^{\infty}\left(\frac{m\coth(\pi m x/y)}{m^2+y^2} - \frac{m\coth(\pi m y/x)}{m^2-x^2}\right) \\ &\quad - 2\pi xy\sum_{m=1}^{\infty}\frac{1}{m}\left(\coth(\pi m x/y) - \coth(\pi m y/x)\right) \\ &= 1 + \frac{\pi^2}{3}(y^2 - x^2)\end{aligned}$$

$$+ 2\pi xy \sum_{m=1}^{\infty} \left(\frac{m \coth(\pi m x/y)}{m^2 + y^2} - \frac{m \coth(\pi m y/x)}{m^2 - x^2} \right)$$
$$- 4\pi xy \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2\pi m x/y} - 1} - \frac{1}{e^{2\pi m y/x} - 1} \right), \tag{12.3.9}$$

where we used the elementary identity

$$\coth x = 1 + \frac{2}{e^{2x} - 1}. \tag{12.3.10}$$

We are now in a position to make simple changes of variables in (12.3.9) to derive a corrected version of (12.3.1).

Entry 12.3.2 (Corrected Version of (21)). *Under the hypotheses of Entry 12.3.1,*

$$\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\}. \tag{12.3.11}$$

Proof. Let $\pi x = \sqrt{w\alpha}$ and $\pi y = \sqrt{w\beta}$ in (12.3.9) to deduce that

$$\begin{aligned}
&\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) \\
&= \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\
&\quad - 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2m\alpha} - 1} - \frac{1}{e^{2m\beta} - 1} \right) \\
&= \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\
&\quad - 2 \left(\frac{1}{4} \log \alpha - \frac{\alpha}{12} - \frac{1}{4} \log \beta + \frac{\beta}{12} \right) \\
&= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\},
\end{aligned}$$

where we have used an equivalent formulation for the transformation of the Dedekind eta function, namely [68],

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \frac{1}{4}\log\alpha + \frac{\alpha}{12} = \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} - \frac{1}{4}\log\beta + \frac{\beta}{12}, \quad (12.3.12)$$

under the condition $\alpha\beta = \pi^2$. This completes the proof of (12.3.11). □

Thus, Ramanujan's claim (21) was correct except for the missing term $\frac{1}{2}\log\frac{\beta}{\alpha}$.

We now proceed to examine the four deductions Ramanujan made from (12.3.1). We first examine the claim that cannot be formally deduced from either (12.3.1) or the corrected version (12.3.11), and provide Ramanujan's argument. Ramanujan asserts that "Equating the coefficients of $1/n$ ($1/w$ in our notation) in both sides in (21) we have ...."

Entry 12.3.3 (p. 318, formula (23)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \quad (12.3.13)$$

Proof. (incorrect) Following Ramanujan, we equate coefficients of $1/w$ on both sides of (12.3.11). Observe from the Laurent expansion of $\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta})$ about $w = 0$ that the coefficient of $1/w$ equals $\frac{1}{2}$ on the left side of (12.3.11). Note also the term $1/(2w)$ on the right side of (12.3.11). Hence, the only contribution of $1/w$ that remains must come from

$$\sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w+m^2\alpha}\left(1+\frac{2}{e^{2m\alpha}-1}\right) + \frac{m\beta}{w-m^2\beta}\left(1+\frac{2}{e^{2m\beta}-1}\right) \right\}, \quad (12.3.14)$$

upon the use of (12.3.10), and this contribution must equal 0.

Proceeding formally, we have

$$\frac{m\alpha}{w+m^2\alpha} = \frac{m\alpha}{w}\sum_{r=0}^{\infty}\left(-\frac{m^2\alpha}{w}\right)^r \qquad \text{and} \qquad \frac{m\beta}{w-m^2\beta} = \frac{m\beta}{w}\sum_{r=0}^{\infty}\left(\frac{m^2\beta}{w}\right)^r.$$

Thus, from (12.3.14) we find that a contribution to the coefficient of $1/w$ equals

$$2\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + 2\beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1}. \quad (12.3.15)$$

The remaining contribution to the coefficient of $1/w$ in (12.3.14) is given by

$$(\alpha+\beta)\sum_{m=1}^{\infty} m = (\alpha+\beta)\zeta(-1) = -\frac{\alpha+\beta}{12}. \quad (12.3.16)$$

Of course, this agrument is not rigorous. The value $\zeta(-1) = -\frac{1}{12}$ can be found in Titchmarsh's book [306, p. 19, Eq. (2.4.3)], for example. Alternatively, the "constant" for the series $\sum_{m=1}^{\infty} m$ in Ramanujan's terminology is equal to $-\frac{1}{12}$ [37, p. 135, Example 2]. Recalling that the contributions of the coefficients of $1/w$ in (12.3.14) must equal 0, we find from (12.3.15) and (12.3.16) that

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1} = \frac{\alpha+\beta}{24}. \tag{12.3.17}$$

In comparing (12.3.17) with (12.3.13), we find that the term $-\frac{1}{4}$ in (12.3.13) does not appear in (12.3.17). This concludes what we think must have been Ramanujan's argument. □

Entry 12.3.4 (pp. 318–319, formula (24)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma(m) = \sum_{d|m} d$, then*

$$\alpha \sum_{m=1}^{\infty} \sigma(m) e^{-2m\alpha} + \beta \sum_{m=1}^{\infty} \sigma(m) e^{-2m\beta} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \tag{12.3.18}$$

Proof. Entry 12.3.4 is simply another version of Entry 12.3.3. To that end, expand the summands of (12.3.13) into geometric series and collect the coefficients of $e^{-2m\alpha}$ and $e^{-2m\beta}$ to complete the proof. □

Ramanujan offered Entry 12.3.3 as Corollary (i) in Sect. 8 of Chap. 14 in his second notebook [268], [38, p. 255]. To the best of our knowledge, Entry 12.3.3 was first proved by O. Schlömilch [279, 280] in 1877. There now exist many proofs; see [38, p. 256] for references to several proofs. One of the most common proofs of the special case $\alpha = \beta = \pi$ of both Entries 12.3.3 and 12.3.6 was recently rediscovered by O. Ogievetsky and V. Schechtman [236]. Entry 12.3.3 is equivalent to the transformation formula for Ramanujan's Eisenstein series $P(q)$.

Entry 12.3.5 (p. 320, formula (29)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma_k(m) = \sum_{d|m} d^k$, then*

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} \\ &\quad = \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\alpha} - \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\beta} = \frac{1}{4}\log\frac{\alpha}{\beta} - \frac{\alpha-\beta}{12}. \end{aligned} \tag{12.3.19}$$

Proof. Following but altering Ramanujan's directions, we equate the terms independent of w in (12.3.11) (not (12.3.1)) and use (12.3.10) to deduce that

$$\begin{aligned} \frac{\pi}{2}\left(-\frac{\sqrt{\alpha}}{3\sqrt{\beta}} + \frac{\sqrt{\beta}}{3\sqrt{\alpha}}\right) &= \frac{1}{2}\log\frac{\beta}{\alpha} \\ &\quad + \sum_{m=1}^{\infty} \left\{ \frac{1}{m}\left(1 + \frac{2}{e^{2m\alpha}-1}\right) - \frac{1}{m}\left(1 + \frac{2}{e^{2m\beta}-1}\right) \right\}. \end{aligned}$$

The desired result (12.3.19) now follows upon simplification, with the use of the identity $\alpha\beta = \pi^2$. ☐

Entry 12.3.5 is stated by Ramanujan as Corollary (ii) in Sect. 8 of Chap. 14 in his second notebook [268], [38, p. 256] and as Entry 27(iii) in Chap. 16 of his second notebook [268], [39, p. 43]. It is equivalent to the transformation formula for the Dedekind eta function. Note that we already used (12.3.19) in the equivalent form (12.3.12) in order to obtain a corrected version of Entry 12.3.1.

The Bernoulli numbers B_m, $m \geq 0$, are defined by

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m, \qquad |z| < 2\pi.$$

This convention for Bernoulli numbers is not the same as that used by Ramanujan in his unpublished manuscript.

Entry 12.3.6 (p. 319, formula (25)). *Let α and β be positive numbers such that $\alpha\beta = \pi^2$, and let B_m, $m \geq 0$, denote the mth Bernoulli number. Then, if r is a positive integer with $r \geq 2$,*

$$\alpha^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1} - \frac{B_{2r}}{4r} \right). \qquad (12.3.20)$$

Proof. (nonrigorous) Return to (12.3.11), use (12.3.10), and formally expand the summands into geometric series to arrive at

$$\begin{aligned} \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) &= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} \\ &+ \sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w} \sum_{k=0}^{\infty} \left(-\frac{m^2\alpha}{w} \right)^k \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) \right. \\ &\left. + \frac{m\beta}{w} \sum_{k=0}^{\infty} \left(\frac{m^2\beta}{w} \right)^k \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}. \end{aligned} \qquad (12.3.21)$$

Following Ramanujan's directions, we equate coefficients of $1/w^r$, $r \geq 2$, on both sides of (12.3.21) to formally deduce that

$$\begin{aligned} 0 = (-1)^{r-1} \alpha^r \zeta(1 - 2r) &+ 2(-1)^{r-1} \alpha^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} \\ &+ \beta^r \zeta(1 - 2r) + 2\beta^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1}. \end{aligned}$$

Using the relation [306, p. 19, Eq. (2.4.3)]

$$\zeta(1 - 2r) = -\frac{B_{2r}}{2r}, \quad r \geq 1,$$

dividing both sides by $2(-1)^r$, and simplifying, we deduce (12.3.20). ☐

Entry 12.3.6 is identical to Entry 13 in Chap. 14 of Ramanujan's second notebook [268], [38, p. 261]. To the best of our knowledge, the first published proof of Entry 12.3.6 was given by M.B. Rao and M.V. Ayyar [271] in 1923. There exist many proofs of Entry 12.3.6, and even more proofs for the special case $\alpha = \beta = \pi$; see [38, pp. 261–262] for references. N.S. Koshliakov [189, 192] has derived interesting analogues of Entry 12.3.6 and other entries in this section.

Expanding the summands in geometric series, we deduce, as in previous entries, the following corollary, which is, in essence, the transformation formula for classical Eisenstein series.

Entry 12.3.7 (p. 319, formula (26)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer with $r \geq 2$, then*

$$\alpha^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\alpha} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\beta} - \frac{B_{2r}}{4r} \right).$$

Entry 12.3.8 (p. 319, formula (27)). *We have*

$$\sum_{m=1}^{\infty} \sigma_5(m) e^{-2\pi m} = \frac{1}{504}.$$

Proof. Entry 12.3.8 follows immediately from Entry 12.3.7 by setting $r = 3$ and $\alpha = \beta = \pi$, and then using the fact that $B_6 = \frac{1}{42}$. □

Entry 12.3.9 (pp. 319–320, formula (28)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer, then*

$$\begin{aligned}
&(4\alpha)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\alpha}-1)} \right) \\
&\quad - (-4\beta)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\beta}-1)} \right) \\
&= (4\alpha)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\alpha} \right) \\
&\quad - (-4\beta)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\beta} \right) \\
&= -\sum_{k=0}^{r+1} \frac{(-1)^k B_{2k} B_{2r+2-2k} \alpha^{r+1-k} \beta^k}{(2k)!(2r+2-2k)!}.
\end{aligned} \tag{12.3.22}$$

Proof. Return to (12.3.11), use (12.3.10), and expand the summands into geometric series to arrive at

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2}\log\frac{\beta}{\alpha} + \sum_{m=1}^{\infty}\left\{\frac{1}{m}\sum_{k=0}^{\infty}\left(-\frac{w}{m^2\alpha}\right)^k\left(1+\frac{2}{e^{2m\alpha}-1}\right) - \frac{1}{m}\sum_{k=0}^{\infty}\left(\frac{w}{m^2\beta}\right)^k\left(1+\frac{2}{e^{2m\beta}-1}\right)\right\}. \tag{12.3.23}$$

Following Ramanujan's advice, we equate coefficients of w^r, $r \geq 1$, on both sides of (12.3.23). On the right side, the coefficient of w^r equals

$$(-\alpha)^{-r}\zeta(2r+1) + 2(-\alpha)^{-r}\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\alpha}-1)} - \beta^{-r}\zeta(2r+1) + 2\beta^{-r}\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\beta}-1)}. \tag{12.3.24}$$

Using the Laurent expansions for $\cot z$ and $\coth z$ about $z=0$, we find that on the left side of (12.3.23),

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}B_{2k}}{(2k)!}(w\alpha)^{k-1/2} \times \sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}(w\beta)^{j-1/2}. \tag{12.3.25}$$

The coefficient of w^r in (12.3.25) is easily seen to be equal to

$$2^{2r+1}\sum_{k=0}^{r+1}\frac{(-1)^k B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!}\alpha^k\beta^{r+1-k}, \tag{12.3.26}$$

where we used the equality $\alpha\beta = \pi^2$. Now equate the expressions in (12.3.24) and (12.3.26), then multiply both sides by $(-1)^r 2^{-2r-1}$, and lastly replace k by $r+1-k$ in the finite sum. We then have shown the equality of the first and third expressions in (12.3.22). The first equality of (12.3.22) follows as before by expanding the summands on the left side into geometric series. □

Entry 12.3.9 is the same as Entry 21(i) in Chap. 14 of Ramanujan's second notebook [268], [38, pp. 275–276]. An extensive generalization of Entry 12.3.9 can be found in Entry 20 of Chap. 16 in Ramanujan's first notebook [268], [40, pp. 429–432]. The special case $\alpha = \beta = \pi$ of Entry 12.3.9 was first established by M. Lerch [215] in 1901, but the general theorem was not proved in print until S.L. Malurkar [220] did so in 1925. Inspired by two papers by E. Grosswald [130, 131], the second author established a proof of Entry 12.3.9, the first claim in Ramanujan's notebooks that the second author had ever examined;

his first paper on Ramanujan's work was the survey paper [30] on Ramanujan's formula for $\zeta(2n+1)$. However, at about the same time, the second author had established another proof of Ramanujan's formula for $\zeta(2n+1)$ as well as a far-ranging generalization [33, Theorem 5.2]. The former paper and the second author's book [38, p. 276] contain a multitude of references for the many proofs and generalizations of Entry 12.3.9. Sitaramachandrarao [289] gave a proof of Entry 12.3.9 based on his partial fraction decomposition (12.3.8), and so his proof is similar to that of Ramanujan. Further proofs and generalizations have been given by D. Bradley [74], L. Vepštas [308], and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [171, 172]. A very engaging proof, in fact of a significant generalization, via Barnes's multiple zeta functions, was devised by Y. Komori, K. Matsumoto, and H. Tsumura [186]. An especially interesting proof, arising out of a very general asymptotic formula, has been devised by M. Katsurada [179]; see also interesting remarks in his paper [180]. A discussion of Ramanujan's formula in conjunction with numerical calculations has been made by B. Ghusayni [122].

The two infinite series on the far left side of (12.3.22) converge very rapidly. If we "ignore" these two series and let r be odd, then we see that $\zeta(2r+1)$ is "almost" a rational multiple of π^{2r+1}. Continuing this line of thought, suppose that we set $\alpha = \pi z$ and $\beta = \pi z$, and now require that z be a root of

$$\sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\pi z}-1)} + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\pi/z}-1)} = 0.$$

Next, multiply both sides of (12.3.22) by $(-1)^r 2^{2r+1}\pi^r z^{r+1}$ and replace k by $r+1-k$ in the finite sum on the far right-hand side. Hence, for such values of z, we deduce that

$$P_k(z) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{k=0}^{r+1} (-1)^k \frac{B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!} z^{2k} = 0. \tag{12.3.27}$$

Accordingly, S. Gun, M.R. Murty, and P. Rath [138] defined the related polynomials

$$R_{2k+1}(z) := \sum_{j=0}^{k+1} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2j}$$

and showed that all of their nonreal roots lie on the unit circle. Murty, C.J. Smyth, and R.J. Wang [230] discovered further properties of these polynomials. In particular, they discovered bounds for their real zeros, and they proved that the largest real zero approaches 2 from above, as $k \to \infty$. M. Lalín and M.D. Rogers [205] studied polynomials that are similar to $R_{2k+1}(z)$ and that are also related to further identities of Ramanujan, and showed that their zeros lie on the unit circle. The study of the polynomials $P_k(z)$ turns out to be more difficult, and in [205], only partial results were obtained. In particular, for $2 \leq k \leq 1{,}000$, the aforementioned authors showed that all of the roots of $P_k(z)$ lie on the unit circle. Finally, Lalín and Smyth [206] proved that all zeros of $P_k(z)$ are indeed located on $|z| = 1$.

12.4 A Correct Partial Fraction Decomposition and Hyperbolic Secant Sums

As in the previous section, we alter Ramanujan's notation by setting $n = w = z^2$.

Entry 12.4.1 (p. 318, formula (22)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if $w \neq -(2m+1)^2\alpha, (2m+1)^2\beta$, $0 \leq m < \infty$, then*

$$\frac{\pi}{4}\sec(\sqrt{w\alpha})\operatorname{sech}(\sqrt{w\beta}) = \sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{w+(2m+1)^2\alpha} - \frac{(2m+1)\beta\operatorname{sech}(2m+1)\beta}{w-(2m+1)^2\beta}\right\}. \tag{12.4.1}$$

Proof. We apply the Mittag–Leffler theorem to

$$f(z) := \frac{\pi}{4}\sec(z\sqrt{\alpha})\operatorname{sech}(z\sqrt{\beta}),$$

which has simple poles at $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = (2m+1)\pi i/(2\sqrt{\beta})$, for each integer m. The residues are easily calculated to be

$$R((2m+1)\pi/(2\sqrt{\alpha})) = -\frac{(-1)^m\pi}{4\sqrt{\alpha}}\operatorname{sech}(2m+1)\beta \tag{12.4.2}$$

and

$$R((2m+1)\pi i/(2\sqrt{\beta})) = \frac{(-1)^m\pi}{4i\sqrt{\beta}}\operatorname{sech}(2m+1)\alpha, \tag{12.4.3}$$

where we used the relation $\alpha\beta = \pi^2/4$. By (12.4.2), the contributions from the poles $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = -(2m+1)\pi/(2\sqrt{\alpha})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\frac{(-1)^m\pi}{4\sqrt{\alpha}}\left(-\frac{\operatorname{sech}(2m+1)\beta}{z-(2m+1)\pi/(2\sqrt{\alpha})} + \frac{\operatorname{sech}(2m+1)\beta}{z+(2m+1)\pi/(2\sqrt{\alpha})}\right) = -\frac{(-1)^m(2m+1)\beta\operatorname{sech}(2m+1)\beta}{z^2-(2m+1)^2\beta}, \tag{12.4.4}$$

where we used the equality $\alpha\beta = \pi^2/4$. Next, by (12.4.3), the contributions of the poles $z = (2m+1)\pi i/(2\sqrt{\beta})$ and $z = -(2m+1)\pi i/(2\sqrt{\beta})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\frac{(-1)^m\pi}{4i\sqrt{\beta}}\left(\frac{\operatorname{sech}(2m+1)\alpha}{z-(2m+1)\pi i/(2\sqrt{\beta})}-\frac{\operatorname{sech}(2m+1)\alpha}{z+(2m+1)\pi i/(2\sqrt{\beta})}\right)$$
$$=\frac{(-1)^m(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{z^2+(2m+1)^2\alpha}, \tag{12.4.5}$$

upon using the equality $\alpha\beta=\pi^2/4$. Thus, applying the Mittag–Leffler theorem and using (12.4.4) and (12.4.5), we find that there exists an entire function $g(z)$ such that

$$\frac{\pi}{4}\sec(z\sqrt{\alpha})\operatorname{sech}(z\sqrt{\beta})=\sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{z^2+(2m+1)^2\alpha}\right.$$
$$\left.-\frac{(2m+1)\beta\operatorname{sech}(2m+1)\beta}{z^2-(2m+1)^2\beta}\right\}+g(z). \tag{12.4.6}$$

Letting $z\to\infty$, we find that $\lim_{z\to\infty}g(z)=0$. Hence, $g(z)\equiv 0$, and thus (12.4.1) follows to complete the proof. □

An equivalent formulation of Entry 12.4.1 is found as Entry 19(iv) in Chap. 14 of Ramanujan's second notebook [268], [38, p. 273], where a different kind of proof was indicated by Ramanujan.

Entry 12.4.2 (p. 320, formula (30)). *If $\alpha\beta=\pi^2/4$, where α and β are positive numbers, and if r is any positive integer, then*

$$\alpha^r\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)^{2r-1}}{\cosh(2m+1)\alpha}+(-\beta)^r\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)^{2r-1}}{\cosh(2m+1)\beta}=0. \tag{12.4.7}$$

Proof. (nonrigorous) Return to (12.4.1) and formally expand the summands on the right side into geometric series to deduce that

$$\frac{\pi}{4}\sec(\sqrt{w\alpha})\operatorname{sech}(\sqrt{w\beta})$$
$$=\sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\alpha}{w}\operatorname{sech}(2m+1)\alpha\sum_{k=0}^{\infty}\left(-\frac{(2m+1)^2\alpha}{w}\right)^k\right.$$
$$\left.-\frac{(2m+1)\beta}{w}\operatorname{sech}(2m+1)\beta\sum_{k=0}^{\infty}\left(\frac{(2m+1)^2\beta}{w}\right)^k\right\}. \tag{12.4.8}$$

Equating coefficients of $1/w^r$, $r\geq 1$, on both sides of (12.4.8), we find that

$$0=\sum_{m=0}^{\infty}(-1)^{m+r-1}(2m+1)^{2r-1}\alpha^r\operatorname{sech}(2m+1)\alpha$$
$$-\sum_{m=0}^{\infty}(-1)^m(2m+1)^{2r-1}\beta^r\operatorname{sech}(2m+1)\beta,$$

which is easily seen to be equivalent to (12.4.7). □

Entry 12.4.2 is Entry 14 of Chap. 14 in Ramanujan's second notebook [268], [38, p. 262], and the first proof known to us was given by Malurkar [220]. See [38, p. 262] for further references and comments.

As with previous theorems, Ramanujan provides an alternative version of Entry 12.4.2 in terms of divisor sums. The details are similar to those above, and so we do not give them, but we remark that careful attention to the signs of the summands should be taken.

Entry 12.4.3 (p. 321, formula (31)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if r is any positive integer, then*

$$\alpha^r \sum_{m=0}^{\infty}(-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\alpha} + (-\beta)^r \sum_{m=0}^{\infty}(-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\beta} = 0.$$

Recall that the Euler numbers E_{2k}, $k \geq 0$, are defined by [126, p. 42, formula 1.411, no. 10]

$$\operatorname{sech} z = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}, \qquad |z| < \pi/2. \tag{12.4.9}$$

Entry 12.4.4 (p. 321, formula (32)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, if r is any positive integer, and if χ denotes the nonprincipal primitive character of modulus 4, as in Sect. 12.2, then*

$$\begin{aligned} &2\alpha^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m) m^{1-2r}}{\cosh(m\alpha)} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m) m^{1-2r}}{\cosh(m\beta)} \\ &\quad = 4\alpha^{1-r} \sum_{m=1}^{\infty} \chi(m)\sigma_{1-2r}(m) e^{-m\alpha} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \chi(m)\sigma_{1-2r}(m) e^{-m\beta} \\ &\quad = \frac{\pi}{2} \sum_{k=0}^{r-1} (-1)^k \frac{E_{2k} E_{2r-2-2k}}{(2k)!(2r-2-2k)!} \alpha^{r-1-k} \beta^k. \end{aligned} \tag{12.4.10}$$

Proof. Return to (12.4.1) and expand both sides in Taylor series about 0. Using (12.4.9), we find that

$$\begin{aligned} &\frac{\pi}{4} \sum_{j=0}^{\infty} (-1)^j \frac{E_{2j}}{(2j)!} (w\alpha)^j \cdot \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} (w\beta)^k \\ &\quad = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\alpha \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2 \alpha} \right)^r \\ &\qquad + \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\beta \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2 \beta} \right)^r. \end{aligned} \tag{12.4.11}$$

In (12.4.11) we equate coefficients of w^{r-1}, $r \geq 1$, on both sides to deduce that

$$\frac{\pi}{4}\sum_{j=0}^{r-1}(-1)^j\frac{E_{2j}E_{2r-2j-2}}{(2j)!(2r-2j-2)!}\alpha^j\beta^{r-j-1}$$
$$=\alpha^{1-r}\sum_{m=0}^{\infty}\frac{(-1)^{m+1-r}\operatorname{sech}(2m+1)\alpha}{(2m+1)^{2r-1}}+\beta^{1-r}\sum_{m=0}^{\infty}\frac{(-1)^{m}\operatorname{sech}(2m+1)\beta}{(2m+1)^{2r-1}}. \tag{12.4.12}$$

Now set $j=r-1-k$ in the sum on the left side of (12.4.12) and multiply both sides of (12.4.12) by $2(-1)^{r-1}$. We then readily deduce the equality of the first and third expressions in (12.4.10). The first equality of (12.4.10) follows as usual from expanding the summands on the left side into geometric series. □

Entry 12.4.4 appears in two formulations, Entries 21(ii), (iii), in Chap. 14 of Ramanujan's second notebook [268], [38, pp. 276–277]. The first proofs of Entry 12.4.4 were found by Malurkar [220] and Chowla [93], [95, pp. 143–170], and further references can be found in [38, p. 277].

Entry 12.4.5 (p. 321, formula (33)). *We have*

$$4\sum_{m=0}^{\infty}(-1)^m\sigma_{-1}(m)e^{-(2m+1)\alpha}+4\sum_{m=0}^{\infty}(-1)^m\sigma_{-1}(m)e^{-(2m+1)\beta}=\frac{\pi}{2}.$$

Proof. Set $r=1$ in Entry 12.4.4. □

S.-G. Lim [216] has generalized many of Ramanujan's theorems on infinite series identities from Ramanujan's notebooks [268], in particular from Chap. 14 in his second notebook, [38, Chap. 14]. For Example, Lim [216, Corollaries 3.33, 3.35] has proved the following two results that generalize Entries 12.3.3 and 12.3.5, respectively. Let α and β be positive numbers such that $\alpha\beta=\pi^2$. Suppose that c is any positive integer. Then

$$\alpha\sum_{m=1}^{\infty}\frac{m}{e^{2m(\alpha-i\pi)/c}-1}+\beta\sum_{m=1}^{\infty}\frac{m}{e^{2m(\beta+i\pi)/c}-1}=\frac{\alpha+\beta}{24}-\frac{c}{4}$$

and

$$\sum_{m=1}^{\infty}\frac{1}{m(e^{2m(\alpha-i\pi)/c}-1)}-\sum_{m=1}^{\infty}\frac{1}{m(e^{2m(\beta+i\pi)/c}-1)}$$
$$=\frac{1}{4}\log\frac{\alpha}{\beta}-\frac{\alpha-\beta}{12c}+\frac{(c-1)(c-2)\pi i}{12c}.$$

When $c=1$ in the identities above, we deduce Entries 12.3.3 and 12.3.5, respectively.

In another paper [217], Lim has found generalizations of the results in Sect. 12.4. We state one of his general theorems.

Theorem 12.4.1. *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Let r be any real number such that $0 < r < 1$. Then, for any integer n,*

$$\begin{aligned}\alpha^{-n}\sum_{k=1}^{\infty}&\frac{(-1)^k\sinh((1-2r)\alpha k)\sin((1-2r)\pi k)}{k^{2n+1}\sinh(\alpha k)}\\&=-(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{(-1)^k\sinh((1-2r)\beta k)\sin((1-2r)\pi k)}{k^{2n+1}\sinh(\beta k)}\\&\quad-2^{2n+1}\pi\sum_{k=0}^{n}\frac{B_{2k+1}(r)B_{2n+1-2k}(r)}{(2k+1)!(2n+1-2k)!}\alpha^{n-k}(-\beta)^k,\end{aligned}\tag{12.4.13}$$

where $B_j(r)$, $j \geq 0$, denotes the jth Bernoulli polynomial.

Although we avoid providing details, setting $r = \frac{1}{4}$ in (12.4.13) yields Entries 12.4.2 and 12.4.4 [217, Corollary 3.23, Proposition 3.21].

13

A Partial Manuscript on Fourier and Laplace Transforms

13.1 Introduction

Pages 219–227 in the volume [269] containing Ramanujan's lost notebook are devoted to material "Copied from the Loose Papers." These "loose papers," in the handwriting of G.N. Watson, are housed in the Oxford University Library, while the original pages in Ramanujan's handwriting, from which the copy was made, are in the library at Trinity College, Cambridge. The three partial manuscripts on these nine pages are in rough form, with two perhaps being drafts of papers being prepared for publication. Most of these nine pages are connected with material in Ramanujan's published papers.

The first manuscript on pages 219–220 is the subject of this chapter. Most of the manuscript is discussed in the next section. Section 13.3 is reserved for the most interesting theorem in the manuscript, namely, a beautiful series transformation involving the logarithmic derivative of the gamma function, which in a second formula, is related to the Riemann zeta function. Our two proofs of this elegant transformation formula are taken from a paper by Berndt and A. Dixit [51]. These two formulas have an interesting history that we relate at the beginning of Sect. 13.3. Since all entries in this chapter can be found on either page 219 or 220 in [269], we refrain from giving page numbers beside entries in the sequel.

13.2 Fourier and Laplace Transforms

Following Ramanujan, we proceed formally without giving attention to such matters as inverting the order of integration in double integrals. It is clear that hypotheses are easily added to make any procedure rigorous.

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Entry 13.2.1. *If*

$$\int_0^\infty f(x)\sin(nx)dx =: \phi(n) \tag{13.2.1}$$

and

$$\int_0^\infty f(x)e^{-nx}dx =: \psi(n), \tag{13.2.2}$$

then

$$\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty \psi(x)\cos(nx)dx \tag{13.2.3}$$

and

$$\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx = -\int_0^\infty \psi\left(\frac{1}{x}\right)\cos(nx)dx. \tag{13.2.4}$$

Proof. We employ the elementary integral evaluations [126, p. 512, Eqs. (3.893), no. 1, no. 2]

$$\int_0^\infty e^{-nx}\sin(xt)dx = \frac{t}{n^2+t^2}, \qquad n>0, \tag{13.2.5}$$

and

$$\int_0^\infty e^{-nx}\cos(xt)dx = \frac{n}{n^2+t^2}, \qquad n>0. \tag{13.2.6}$$

To prove (13.2.3), we use (13.2.1), (13.2.5), (13.2.6), and (13.2.2) to deduce that

$$\begin{aligned}\int_0^\infty \phi(x)e^{-nx}dx &= \int_0^\infty\int_0^\infty f(t)e^{-nx}\sin(xt)\,dt\,dx\\ &= \int_0^\infty f(t)\int_0^\infty e^{-nx}\sin(xt)\,dx\,dt\\ &= \int_0^\infty f(t)\int_0^\infty e^{-tx}\cos(nx)\,dx\,dt\\ &= \int_0^\infty \psi(x)\cos(nx)dx,\end{aligned}$$

which completes the proof of the first claim.

Using (13.2.1) and making the substitution $t=ux$, we find that

$$\begin{aligned}\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx &= \int_0^\infty\int_0^\infty f(t)e^{-nx}\sin(t/x)\,dt\,dx\\ &= \int_0^\infty\int_0^\infty xf(ux)e^{-nx}\sin u\,du\,dx\\ &= -\frac{d}{dn}\int_0^\infty\int_0^\infty f(ux)e^{-nx}\sin u\,dx\,du. \end{aligned}\tag{13.2.7}$$

Note that upon the replacement of n by n/t and x by tu in (13.2.2),

$$\psi\left(\frac{n}{t}\right) = t\int_0^\infty f(tu)e^{-nu}du. \tag{13.2.8}$$

Thus, from (13.2.7) and (13.2.8),

$$\begin{aligned}\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx &= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{n}{u}\right)\frac{\sin u}{u}du\\ &= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{1}{x}\right)\frac{\sin(nx)}{x}dx\\ &= -\int_0^\infty \psi\left(\frac{1}{x}\right)\cos(nx)dx,\end{aligned}$$

which completes the proof of (13.2.4). □

Entry 13.2.2. *If*

$$\int_0^\infty f(x)\cos(nx)dx =: \phi(n) \tag{13.2.9}$$

and

$$\int_0^\infty f(x)e^{-nx}dx =: \psi(n), \tag{13.2.10}$$

then

$$\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty \psi(x)\sin(nx)dx \tag{13.2.11}$$

and

$$\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx = \int_0^\infty \psi\left(\frac{1}{x}\right)\sin(nx)dx. \tag{13.2.12}$$

Proof. The details of the proof of Entry 13.2.2 are completely analogous to those for the proof of Entry 13.2.1, and so there is no need to give them here. □

Suppose now that $f(x)$ is self-reciprocal in Entries 13.2.1 and 13.2.2, that is to say,

$$f(x) = \sqrt{\frac{2}{\pi}}\phi(x).$$

Hence, from (13.2.2),

$$\int_0^\infty \phi(x)e^{-nx}dx = \sqrt{\frac{\pi}{2}}\int_0^\infty f(x)e^{-nx}dx = \sqrt{\frac{\pi}{2}}\psi(n).$$

Then we see that (13.2.3) and (13.2.11) easily yield the next theorem.

Entry 13.2.3. *If*

$$\int_0^\infty \phi(x)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\phi(n)$$

and

$$\int_0^\infty \phi(x)e^{-nx}dx =: \psi(n),$$

then

$$\int_0^\infty \psi(x)\cos(nx)dx = \sqrt{\frac{\pi}{2}}\psi(n). \tag{13.2.13}$$

If

$$\int_0^\infty \phi(x)\cos(nx)dx = \sqrt{\frac{\pi}{2}}\phi(n)$$

and

$$\int_0^\infty \phi(x)e^{-nx}dx =: \psi(n),$$

then

$$\int_0^\infty \psi(x)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\psi(n). \tag{13.2.14}$$

Ramanujan then writes that (13.2.13) and (13.2.14) "enable us to find a number of reciprocal functions of the first and second kind out of one reciprocal function." He does not define what he means by "the first and second kind." Some examples of self-reciprocal functions are next recorded.

Entry 13.2.4. *For $n > 0$,*

$$\int_0^\infty \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)\sin(2\pi nx)dx = \frac{1}{2}\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right). \tag{13.2.15}$$

Proof. This result is well known, and we shall be content with quoting from Titchmarsh's *Theory of Fourier Integrals* [305, p. 245]:

$$\begin{aligned}\frac{1}{e^{\sqrt{2\pi}\,x}-1} - \frac{1}{\sqrt{2\pi}\,x} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,y}-1} - \frac{1}{\sqrt{2\pi}\,y}\right)\sin(xy)dy \\ &= 2\int_0^\infty \left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)\sin(\sqrt{2\pi}\,xu)du.\end{aligned}$$

Replacing x by $\sqrt{2\pi}\,n$, we immediately verify Ramanujan's claim. □

It will be convenient to use the familiar notation [126, p. 952, formulas 8.360, 8.362, no. 1]

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right), \tag{13.2.16}$$

where γ denotes Euler's constant. The notation $\psi(x)$ conflicts with the generic notation that we have utilized in Entries 13.2.1 and 13.2.2, but no confusion should arise in the sequel.

Entry 13.2.5. *For $n > 0$,*

$$\int_0^{\infty}\left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)e^{-2\pi n x}dx = \frac{1}{2\pi}\left(\log n - \psi(1+n)\right). \tag{13.2.17}$$

In the manuscript in [269], a factor of $-1/(2\pi)$ is missing on the right-hand side of (13.2.17).

Proof. We begin with the evaluation [126, p. 377, formula 3.427, no. 7]

$$\int_0^{\infty}\left(\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu x}}{x}\right)dx = \log\mu - \psi(\nu),$$

where $\mu, \nu > 0$. Set $\nu = n+1$ and $\mu = n$ to deduce, after simplification, that

$$\begin{aligned}\log n - \psi(n+1) &= \int_0^{\infty}\left(\frac{1}{e^{x}-1} - \frac{1}{x}\right)e^{-nx}dx \\ &= 2\pi\int_0^{\infty}\left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)e^{-2\pi n u}du.\end{aligned}$$

Thus, (13.2.17) is apparent. □

Entry 13.2.6. *If $n > 0$,*

$$\int_0^{\infty}\left(\psi(1+x) - \log x\right)\cos(2\pi n x)dx = \frac{1}{2}\left(\psi(1+n) - \log n\right). \tag{13.2.18}$$

Proof. Setting $u = 2\pi x$ in (13.2.15), we record that

$$\int_0^{\infty}\left(\frac{1}{e^{u}-1} - \frac{1}{u}\right)\sin(nu)du = \pi\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right). \tag{13.2.19}$$

Thus, in the notation of Entry 13.2.1,

$$f(x) = \frac{1}{e^{x}-1} - \frac{1}{x} \qquad \text{and} \qquad \phi(n) = \pi\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right).$$

By (13.2.17),

$$\begin{aligned}\int_0^{\infty} f(x)e^{-nx}dx &= \int_0^{\infty}\left(\frac{1}{e^{x}-1} - \frac{1}{x}\right)e^{-nx}dx \\ &= 2\pi\int_0^{\infty}\left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)e^{-2\pi n u}du \\ &= \log n - \psi(1+n). \end{aligned} \tag{13.2.20}$$

Thus, in the notation of Entry 13.2.1,

$$\begin{aligned}\int_0^\infty \phi(x)e^{-nx}dx &= \pi\int_0^\infty \left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)e^{-nx}dx\\ &= \int_0^\infty (\log x-\psi(1+x))\cos(nx)dx.\end{aligned}$$

Replacing n by $2\pi n$ above, we find that

$$\pi\int_0^\infty \left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)e^{-2\pi nx}dx = \int_0^\infty (\log x-\psi(1+x))\cos(2\pi nx)dx,$$

or, by (13.2.20),

$$\frac{1}{2}(\log n-\psi(1+n)) = \int_0^\infty (\log x-\psi(1+x))\cos(2\pi nx)dx,$$

as claimed. □

Ramanujan next quotes the following self-reciprocal Fourier cosine transform [126, p. 537, formula 3.981, no. 3].

Entry 13.2.7. *For real* n*,*

$$\int_0^\infty \frac{\cos(\frac{1}{2}\pi nx)}{\cosh(\frac{1}{2}\pi x)}dx = \frac{1}{\cosh(\frac{1}{2}\pi n)}. \tag{13.2.21}$$

Then he records the following entry.

Entry 13.2.8. *For* $n>0$*,*

$$\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx = \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1}. \tag{13.2.22}$$

This follows from the evaluation [126, p. 399, formula 3.541, no. 6]

$$\begin{aligned}\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx &= \frac{1}{\pi}\left\{\psi\left(\frac{n+3}{4}\right)-\psi\left(\frac{n+1}{4}\right)\right\}\\ &= \frac{4}{\pi}\sum_{k=0}^\infty\left(-\frac{1}{4k+n+3}+\frac{1}{4k+n+1}\right)\\ &= \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{2k+n+1},\end{aligned}$$

where we utilized (13.2.16).

Entry 13.2.9. *For $n > 0$,*

$$\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1} \sin(\tfrac{1}{2}\pi nx)dx = \sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1}. \tag{13.2.23}$$

Proof. Rewrite (13.2.21) as

$$\int_0^\infty \frac{\cos(nu)}{\cosh u} du = \frac{\pi}{2\cosh(\frac{1}{2}\pi n)}.$$

We are thus going to apply Entry 13.2.2 with

$$f(x) = \frac{1}{\cosh x} \qquad \text{and} \qquad \phi(x) = \frac{\pi}{2\cosh(\frac{1}{2}\pi x)}.$$

From (13.2.22),

$$\begin{aligned}\int_0^\infty f(x)e^{-nx}dx = \int_0^\infty \frac{e^{-nx}}{\cosh x}dx &= \frac{\pi}{2}\int_0^\infty \frac{e^{-\frac{1}{2}\pi nu}}{\cosh(\frac{1}{2}\pi u)}du \\ &= 2\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1} := \psi(n).\end{aligned}$$

Hence, by Entry 13.2.2,

$$\frac{\pi}{2}\int_0^\infty \frac{e^{-nx}}{\cosh(\frac{1}{2}\pi x)}dx = 2\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(nx)dx, \tag{13.2.24}$$

or, if we replace n by $\frac{1}{2}\pi n$,

$$\frac{\pi}{4}\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx = \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx.$$

Lastly, if we employ (13.2.22) in the foregoing equality, we conclude that

$$\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1} = \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx, \tag{13.2.25}$$

which is what we wanted to prove. □

Next Ramanujan restates Entries 13.2.1 and 13.2.2 under the assumption

$$f(x) = \sqrt{\frac{2}{\pi}}\phi(x),$$

that is to say, $\phi(x)$ is self-reciprocal. Since his claims are identical to those in Entry 13.2.3, we forego restating them here.

Ramanujan then provides some examples, which are essentially ones that he gave above. First,

$$\int_0^\infty \frac{\cos(nx)}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}dx = \sqrt{\frac{\pi}{2}}\frac{1}{\cosh\left(n\sqrt{\frac{\pi}{2}}\right)},$$

which is an easy consequence of (13.2.21). Second,

$$\int_0^\infty \frac{e^{-nx}}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}dx = 2\sum_{k=0}^\infty \frac{(-1)^k}{n+\sqrt{\frac{\pi}{2}}(2k+1)}.$$

To establish this identity, replace x by $\sqrt{2/\pi}\,x$ and n by $\sqrt{2/\pi}\,n$ in (13.2.22). Third,

$$\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx = \sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1},$$

which is the same as (13.2.25). Fourth,

$$\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,x}-1}-\frac{1}{\sqrt{2\pi}\,x}\right)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\left(\frac{1}{e^{\sqrt{2\pi}\,n}-1}-\frac{1}{\sqrt{2\pi}\,n}\right).$$

This last identity follows easily from (13.2.15) upon replacing x by $x/\sqrt{2\pi}$ and n by $n/\sqrt{2\pi}$.

The next two examples contain errors. Ramanujan's fifth example asserts that

$$\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,x}-1}-\frac{1}{\sqrt{2\pi}\,x}\right)e^{-nx}dx = \sqrt{2\pi}\left\{\gamma+\log\frac{n}{\sqrt{2\pi}}-\psi\left(1+\frac{n}{\sqrt{2\pi}}\right)\right\}, \tag{13.2.26}$$

where γ denotes Euler's constant and $\psi(x)$ is defined in (13.2.16). Return to (13.2.17) and replace x by $x/\sqrt{2\pi}$ and n by $n/\sqrt{2\pi}$. Because, as we previously noted, Ramanujan missed a factor of $-1/(2\pi)$ in (13.2.17), we see that the factor $\sqrt{2\pi}$ on the right-hand side above should be replaced by $-1/\sqrt{2\pi}$. However, there is another error in (13.2.26), because of the spurious appearance of γ on the right-hand side of (13.2.26). Lastly, Ramanujan asserts that

$$\int_0^\infty \{\gamma+\log x-\psi(1+x)\}\cos(2\pi nx)dx = \frac{1}{2}\{\gamma+\log n-\psi(1+n)\}. \tag{13.2.27}$$

To see that the claim (13.2.27) is false, we recall that [1, p. 259], as $x\to\infty$,

$$\psi(x+1) \sim \log x + \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{13.2.28}$$

Thus, we see that the integral in (13.2.27) diverges.

13.3 A Transformation Formula

The most interesting claim made by Ramanujan in the fragment on pages 219 and 220 of [269] is the next entry. To state this claim, we need to recall the following functions associated with Riemann's zeta function $\zeta(s)$. Let

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it). \tag{13.3.1}$$

Entry 13.3.1. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{13.3.2}$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\begin{aligned}\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\phi(n\alpha)\right\} &= \sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{n=1}^{\infty}\phi(n\beta)\right\}\\ &= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \end{aligned} \tag{13.3.3}$$

where γ denotes Euler's constant and $\Xi(x)$ denotes Riemann's Ξ-function.

Although Ramanujan does not provide a proof of (13.3.3), he does indicate that (13.3.3) "can be deduced from" Entry 13.2.6, or (13.2.18). This remark might lead one to believe that his proof of (13.3.3) rests upon the Poisson summation formula. We provide below a proof of the first equality in (13.3.3) that naturally establishes the second equality as well. Then we give a proof of the first equality in (13.3.3) by means of the Poisson summation formula, but, as we indicated, no connection with $\zeta(s)$ and the integral in the second equality is obtained in this way. In both proofs, the self-reciprocal Fourier cosine transform in (13.2.18) is an essential ingredient.

The self-reciprocal property of $\psi(1+x) - \log x$ was rediscovered by A.P. Guinand [133] in 1947, and he later found a simpler proof of this result in [135]. In a footnote at the end of his paper [135], Guinand remarks that T.A. Brown had told him that he himself had proved the self-reciprocality of $\psi(1+x) - \log x$ some years ago, and that when he (Brown) communicated the result to G.H. Hardy, Hardy told him that the result was also given by Ramanujan in a progress report to the University of Madras, but was not published elsewhere. However, we cannot find this result in any of the three *Quarterly Reports* that Ramanujan submitted to the University of Madras [35–37]. In contrast to what Hardy recalled, it would appear that he saw (13.2.18) in the aforementioned manuscript that Watson had copied. We surmise that Hardy once possessed the original copies of both the *Quarterly Reports* and the present manuscript on pages 219–220 of [269], both of which were most likely mailed to him on August 30, 1923, by Francis Dewsbury, registrar at the University of Madras [64, p. 266]. It could be that the two documents were kept together, and so it is understandable that Hardy concluded that the manuscript was part of the *Quarterly Reports*. Unfortunately, the only copy of Ramanujan's *Quarterly Reports* that now exists is in Watson's handwriting.

The first equality in (13.3.3) was rediscovered by Guinand in [133] and appears in a footnote on the last page of his paper [133, p. 18]. It is interesting that Guinand remarks, "This formula also seems to have been overlooked." Here then is one more instance in which a mathematician thought that his or her theorem was new, but unbeknownst to the claimant, Ramanujan had beaten her/him to the punch! We now give Guinand's version of (13.3.3).

Theorem 13.3.1. *For any complex z such that $|\arg z| < \pi$, we have*

$$\sum_{n=1}^{\infty}\left(\psi(nz) - \log nz + \frac{1}{2nz}\right) + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z}\sum_{n=1}^{\infty}\left(\psi\left(\frac{n}{z}\right) - \log\frac{n}{z} + \frac{z}{2n}\right) + \frac{1}{2}\left(\gamma - \log\frac{2\pi}{z}\right). \quad (13.3.4)$$

The first equality in (13.3.3) can be easily obtained from Guinand's version by multiplying both sides of (13.3.4) by $\sqrt{z}$ and then letting $z = \alpha$ and $1/z = \beta$. Although not offering a proof of (13.3.4) in [133], Guinand did remark that it can be obtained by using an appropriate form of Poisson's summation formula, namely the form given in Theorem 1 in [132]. Later Guinand gave another proof of Theorem 13.3.1 in [135], while also giving extensions of (13.3.4) involving derivatives of the ψ-function. He also established a finite version of (13.3.4) in [137]. However, Guinand apparently did not discover the connection of his work with Ramanujan's integral involving Riemann's Ξ-function.

We first provide a proof of both identities in Entry 13.3.1. Then we construct a second proof of the first equality in (13.3.3), or, more precisely, of (13.3.4), along the lines suggested by Guinand in [133]. We could have also provided another proof of (13.3.3) employing both (13.2.18) and (13.2.17), but this proof is similar but slightly more complicated than the first proof that we provide below. The two proofs of Entry 13.3.1 given here are from a paper by

A. Dixit and the second author [51]. In two further papers [107, 108], Dixit has found further proofs of Entry 13.3.1.

Although the Riemann zeta function appears at various instances throughout Ramanujan's notebooks [268] and lost notebook [269], he wrote only one paper in which the zeta function plays the leading role [257], [267, pp. 72–77]. In fact, a result proved by Ramanujan in [257], namely Eq. (13.3.18) below, is a key to proving (13.3.3). About the integral involving Riemann's Ξ-function in this result, Hardy [143] comments that "the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in the *Acta Mathematica*, to prove that

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+ti\right)\right|^2 dt \sim 2T\log T.\text{"} \tag{13.3.5}$$

(We have corrected a misprint in Hardy's version of (13.3.5).)

In a paper immediately following Ramanujan's paper [257], Hardy [143] remarks that the integral on the right-hand side in Ramanujan's formula [257, p. 75, Eq. (13)] can be used to prove that there are infinitely many zeros of $\zeta(s)$ on the critical line $\operatorname{Re} s = \frac{1}{2}$, and then he concludes his note by stating (13.3.6) below, which he says is not unlike the aforementioned formula of Ramanujan. However, Hardy does not give a proof of his formula. Proofs were independently supplied by N.S. Koshliakov [190],[193, Eq. (20)], [194, Chap. 9, Sect. 36], [196, Eq. (34.10)] and Dixit [107]. In Hardy's formulation, the sign of $\frac{1}{2}\gamma$ should be $+$ and not $-$. The sign error was corrected in the papers by Koshliakov and Dixit, but there is an erroneous added factor of $\log 2$ in Koshliakov's formulation in [196]. Koshliakov [190, 195] and Dixit [111] also have given generalizations of Hardy's result.

Theorem 13.3.2 (Correct version). *For real n,*

$$\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt = \frac{1}{4}e^{-n}\left(2n+\frac{1}{2}\gamma+\frac{1}{2}\log\pi+\log 2\right) + \frac{1}{2}e^{n}\int_0^\infty \psi(x+1)e^{-\pi x^2 e^{4n}}\,dx. \tag{13.3.6}$$

Inexplicably, this short note [143] is not reproduced in any of the seven volumes of the *Collected Papers of G.H. Hardy*!

First Proof of Entry 13.3.1. We first collect several well-known theorems that we use in our proof. First, from [99, p. 191], for $t \neq 0$,

$$\sum_{n=1}^\infty \frac{1}{t^2+4n^2\pi^2} = \frac{1}{2t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right). \tag{13.3.7}$$

Second, from [315, p. 251], we find that for $\operatorname{Re} z > 0$,

$$\phi(z) = -2\int_0^\infty \frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)}. \tag{13.3.8}$$

Third, we require Binet's integral for $\log \Gamma(z)$, i.e., for $\operatorname{Re} z > 0$ [315, p. 249], [126, p. 377, formula 3.427, no. 4],

$$\log \Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}dt. \tag{13.3.9}$$

Fourth, from [126, p. 377, formula 3.427, no. 2], we find that

$$\int_0^\infty \left(\frac{1}{1-e^{-x}}-\frac{1}{x}\right)e^{-x}dx = \gamma, \tag{13.3.10}$$

where γ denotes Euler's constant. Fifth, by Frullani's integral [126, p. 378, formula 3.434, no. 2],

$$\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{x}dx = \log\frac{\nu}{\mu}. \tag{13.3.11}$$

Our first goal is to establish an integral representation for the far left side of (13.3.3). Replacing z by $n\alpha$ in (13.3.8) and summing on n, $1 \le n < \infty$, we find that

$$\begin{aligned}\sum_{n=1}^\infty \phi(n\alpha) &= -2\sum_{n=1}^\infty \int_0^\infty \frac{t\,dt}{(t^2+n^2\alpha^2)(e^{2\pi t}-1)}\\ &= -\frac{2}{\alpha^2}\int_0^\infty \frac{t}{(e^{2\pi t}-1)}\sum_{n=1}^\infty \frac{1}{(t/\alpha)^2+n^2}.\end{aligned} \tag{13.3.12}$$

Invoking (13.3.7) in (13.3.12), we see that

$$\sum_{n=1}^\infty \phi(n\alpha) = -\frac{2\pi}{\alpha}\int_0^\infty \frac{1}{(e^{2\pi t}-1)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt. \tag{13.3.13}$$

Next, setting $x = 2\pi t$ in (13.3.10), we readily find that

$$\gamma = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-2\pi t}}{t}\right)dt. \tag{13.3.14}$$

By Frullani's integral (13.3.11),

$$\int_0^\infty \frac{e^{-t/\alpha}-e^{-2\pi t}}{t}dt = \log\left(\frac{2\pi}{1/\alpha}\right) = \log(2\pi\alpha). \tag{13.3.15}$$

Combining (13.3.14) and (13.3.15), we arrive at

$$\gamma - \log(2\pi\alpha) = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t} - 1} - \frac{e^{-t/\alpha}}{t} \right) dt. \tag{13.3.16}$$

Hence, from (13.3.13) and (13.3.16), we deduce that

$$\begin{aligned} &\sqrt{\alpha} \left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^\infty \phi(n\alpha) \right) \qquad (13.3.17) \\ &= \frac{1}{2\sqrt{\alpha}} \int_0^\infty \left(\frac{2\pi}{e^{2\pi t} - 1} - \frac{e^{-t/\alpha}}{t} \right) dt \\ &\quad - \frac{2\pi}{\sqrt{\alpha}} \int_0^\infty \frac{1}{(e^{2\pi t} - 1)} \left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t} + \frac{1}{2} \right) dt \\ &= \int_0^\infty \left(\frac{\sqrt{\alpha}}{t(e^{2\pi t} - 1)} - \frac{2\pi}{\sqrt{\alpha}(e^{2\pi t/\alpha} - 1)(e^{2\pi t} - 1)} - \frac{e^{-t/\alpha}}{2t\sqrt{\alpha}} \right) dt. \end{aligned}$$

Now from [257, p. 260, Eq. (22)] or [267, p. 77], for n real,

$$\begin{aligned} &\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right) \Gamma\left(\frac{-1-it}{4}\right) \left(\Xi\left(\frac{1}{2}t\right)\right)^2 \frac{\cos nt}{1+t^2}\, dt \\ &= \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos nt}{1+t^2}\, dt \\ &= \pi^{3/2} \int_0^\infty \left(\frac{1}{e^{xe^n} - 1} - \frac{1}{xe^n} \right) \left(\frac{1}{e^{xe^{-n}} - 1} - \frac{1}{xe^{-n}} \right) dx. \qquad (13.3.18) \end{aligned}$$

Letting $n = \frac{1}{2}\log\alpha$ and $x = 2\pi t/\sqrt{\alpha}$ in (13.3.18), we deduce that

$$\begin{aligned} &-\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos(\frac{1}{2}t\log\alpha)}{1+t^2} dt \qquad (13.3.19) \\ &= -\frac{2\pi}{\sqrt{\alpha}} \int_0^\infty \left(\frac{1}{e^{2\pi t} - 1} - \frac{1}{2\pi t} \right) \left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t} \right) dt \\ &= \int_0^\infty \left(\frac{-2\pi/\sqrt{\alpha}}{(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)} + \frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)} + \frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha} - 1)} - \frac{\sqrt{\alpha}}{2\pi t^2} \right) dt. \end{aligned}$$

Hence, combining (13.3.17) and (13.3.19), in order to prove that the far left side of (13.3.3) equals the far right side of (13.3.3), we see that it suffices to show that

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha} - 1)} - \frac{\sqrt{\alpha}}{2\pi t^2} + \frac{e^{-t/\alpha}}{2t\sqrt{\alpha}} \right) dt \\ &\qquad = \frac{1}{\sqrt{\alpha}} \int_0^\infty \left(\frac{1}{u(e^u - 1)} - \frac{1}{u^2} + \frac{e^{-u/(2\pi)}}{2u} \right) du = 0, \qquad (13.3.20) \end{aligned}$$

where we made the change of variable $u = 2\pi t/\alpha$. In fact, more generally, we show that

$$\int_0^\infty \left(\frac{1}{u(e^u - 1)} - \frac{1}{u^2} + \frac{e^{-ua}}{2u} \right) du = -\frac{1}{2}\log(2\pi a), \tag{13.3.21}$$

so that if we set $a = 1/(2\pi)$ in (13.3.21), we deduce (13.3.20).

Consider the integral, for $t > 0$,

$$\begin{aligned} F(a,t) :&= \int_0^\infty \left\{ \left(\frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-tu}}{u} + \frac{e^{-ua} - e^{-tu}}{2u} \right\} du \\ &= \log \Gamma(t) - \left(t - \frac{1}{2}\right) \log t + t - \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\frac{t}{a}, \end{aligned} \tag{13.3.22}$$

where we applied (13.3.9) and (13.3.11). Upon the integration of (13.2.16), it is easily gleaned that as $t \to 0^+$,

$$\log \Gamma(t) \sim -\log t - \gamma t,$$

where γ denotes Euler's constant. Using this in (13.3.22), we find, upon simplification, that as $t \to 0^+$,

$$F(a,t) \sim -\gamma t - t\log t + t - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log a.$$

Hence,

$$\lim_{t\to 0^+} F(a,t) = -\frac{1}{2}\log(2\pi a). \tag{13.3.23}$$

Letting t approach 0^+ in (13.3.22), taking the limit under the integral sign on the right-hand side using Lebesgue's dominated convergence theorem, and employing (13.3.23), we immediately deduce (13.3.21). As previously discussed, this is sufficient to prove the equality of the first and third expressions in (13.3.3), namely,

$$\begin{aligned} &\sqrt{\alpha}\left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^\infty \phi(n\alpha) \right\} \\ &\quad = -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt. \end{aligned} \tag{13.3.24}$$

Lastly, using (13.3.24) with α replaced by β and employing the relation $\alpha\beta = 1$, we conclude that

$$\begin{aligned} &\sqrt{\beta}\left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^\infty \phi(n\beta) \right\} \\ &\quad = -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2} dt \end{aligned}$$

$$= -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t \log(1/\alpha)\right)}{1+t^2} dt$$
$$= -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t \log \alpha\right)}{1+t^2} dt.$$

Hence, the equality of the second and third expressions in (13.3.3) has been demonstrated, and so the proof is complete. □

We next give our second proof of the first identity in (13.3.3) using Guinand's generalization of Poisson's summation formula in [132]. We emphasize that this route does not take us to the integral involving Riemann's Ξ-function in the second identity of (13.3.3). First, we reproduce the needed version of the Poisson summation formula from Theorem 1 in [132].

Theorem 13.3.3. *If $f(x)$ has a Fourier integral representation, $f(x)$ tends to zero as $x \to \infty$, and $xf'(x)$ belongs to $L^p(0,\infty)$, for some p, $1 < p \le 2$, then*

$$\lim_{N\to\infty} \left(\sum_{n=1}^N f(n) - \int_0^N f(t)\, dt \right) = \lim_{N\to\infty} \left(\sum_{n=1}^N g(n) - \int_0^N g(t)\, dt \right),$$

where

$$g(x) = 2\int_0^\infty f(t) \cos(2\pi x t)\, dt. \tag{13.3.25}$$

We first state a lemma[1] that will subsequently be used in our proof of (13.3.3).

Lemma 13.3.1. *If $\psi(x)$ is defined by (13.2.16), then*

$$\int_0^\infty \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt = \frac{1}{2} \log 2\pi. \tag{13.3.26}$$

Proof. Let I denote the integral on the left-hand side of (13.3.26). Then,

$$\begin{aligned} I &= \int_0^\infty \frac{d}{dt} \left(\log \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \right) dt \\ &= \lim_{t\to\infty} \log \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} - \lim_{t\to 0} \log \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \\ &= \log \lim_{t\to\infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} - \log \left(\lim_{t\to 0} e^t \Gamma(t+1) \right) - \lim_{t\to 0} t \log t - \lim_{t\to 0} \frac{1}{2} \log(t+1) \\ &= \log \lim_{t\to\infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}}. \end{aligned} \tag{13.3.27}$$

[1] The authors are indebted to M.L. Glasser for the proof of this lemma. The authors' original proof of this lemma was substantially longer than Glasser's given here.

Next, Stirling's formula [126, p. 945, formula 8.327] tells us that

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-1/2} e^{-z}, \tag{13.3.28}$$

as $|z| \to \infty$ for $|\arg z| \le \pi - \delta$, where $0 < \delta < \pi$. Hence, employing (13.3.28), we find that

$$\frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \sim \frac{\sqrt{2\pi}}{e} \left(1 + \frac{1}{t}\right)^t, \tag{13.3.29}$$

so that

$$\lim_{t\to\infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} = \sqrt{2\pi}. \tag{13.3.30}$$

Thus, from (13.3.27) and (13.3.30), we conclude that

$$I = \frac{1}{2} \log 2\pi. \tag{13.3.31}$$

□

Second Proof of the first equality of (13.3.3)*, or of* (13.3.4)*.* We first prove (13.3.3) for $\operatorname{Re} z > 0$. Let

$$f(x) = \psi(xz+1) - \log xz. \tag{13.3.32}$$

We show that $f(x)$ satisfies the hypotheses of Theorem 13.3.3. From (13.2.18), we see that $f(x)$ has the required integral representation. Next, we need two formulas for $\psi(x)$. First, from [1, p. 259, formula 6.3.18], for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots. \tag{13.3.33}$$

Second, from [315, p. 250],

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{13.3.34}$$

Using the easily verified equality

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{13.3.35}$$

(13.3.32), and (13.3.33), we see that

$$f(x) \sim \frac{1}{2xz} - \frac{1}{12x^2z^2} + \frac{1}{120x^4z^4} - \frac{1}{252x^6z^6} + \cdots, \tag{13.3.36}$$

so that

$$\lim_{x\to\infty} f(x) = 0. \tag{13.3.37}$$

Next, we show that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$. Using (13.3.36), we find that as $x \to \infty$,

$$xf^{'}(x) \sim -\frac{1}{2xz}, \tag{13.3.38}$$

so that $|xf^{'}(x)|^p \sim (2x|z|)^{-p}$. Thus, for $p > 1$, we see that $xf^{'}(x)$ is locally integrable near ∞. Also, using (13.3.35) and (13.3.34), we have

$$\begin{aligned}\lim_{x\to 0} xf^{'}(x) &= \lim_{x\to 0}\left(xz\sum_{n=0}^{\infty}\frac{1}{(xz+n)^2} - \frac{1}{xz} - 1\right)\\ &= \lim_{x\to 0}\left(xz\sum_{n=1}^{\infty}\frac{1}{(xz+n)^2} - 1\right)\\ &= -1. \end{aligned}\tag{13.3.39}$$

This proves that $xf^{'}(x)$ is locally integrable near 0. Hence, we have shown that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$.

Now from (13.3.25) and (13.3.32), we find that

$$g(x) = 2\int_0^{\infty}(\psi(tz+1) - \log tz)\cos(2\pi xt)\,dt.$$

Employing the change of variable $y = tz$ and using (13.2.18), we find that

$$\begin{aligned} g(x) &= \frac{2}{z}\int_0^{\infty}(\psi(y+1) - \log y)\cos(2\pi xy/z)\,dy\\ &= \frac{1}{z}\left(\psi\left(\frac{x}{z}+1\right) - \log\left(\frac{x}{z}\right)\right). \end{aligned}\tag{13.3.40}$$

Substituting the expressions for $f(x)$ and $g(x)$ from (13.3.32) and (13.3.40), respectively, in Theorem 13.3.3, we find that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}(\psi(nz+1) - \log nz) - \int_0^N(\psi(tz+1) - \log tz)\,dt\right) \tag{13.3.41}$$
$$= \frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\psi\left(\frac{n}{z}+1\right) - \log\frac{n}{z}\right) - \int_0^N\left(\psi\left(\frac{t}{z}+1\right) - \log\frac{t}{z}\right)dt\right)\right].$$

Thus, with the use of (13.3.35),

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}(nz)+\frac{1}{2nz}-\log nz\right)\right.$$
$$\left.+\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right)+\frac{z}{2n}-\log\frac{n}{z}\right)\right.\right.$$
$$\left.\left.+\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)\,dt\right)\right]. \tag{13.3.42}$$

Now if we can show that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)=\frac{\gamma-\log 2\pi z}{2z}, \tag{13.3.43}$$

then replacing z by $1/z$ in (13.3.43) will give us

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)\,dt\right)=\frac{z(\gamma-\log(2\pi/z))}{2}. \tag{13.3.44}$$

Then substituting (13.3.43) and (13.3.44) in (13.3.42) will complete the proof of Theorem 13.3.1. To that end,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\lim_{N\to\infty}\left(\frac{1}{2z}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)+\frac{\log N}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{\gamma}{2z}+\lim_{N\to\infty}\left(-\frac{\log z}{2z}+\frac{\log Nz}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2z}-\frac{1}{z}\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right.$$
$$\left.-\frac{1}{2z}\log\left(1+\frac{1}{Nz}\right)\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2}-\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{1}{2}\int_0^{Nz}\frac{1}{t+1}\,dt-\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right)$$

$$
\begin{aligned}
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z} \lim_{N\to\infty} \int_0^{NZ} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z} \int_0^{\infty} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{\log 2\pi}{2z} \\
&= \frac{\gamma - \log 2\pi z}{2z}, \qquad (13.3.45)
\end{aligned}
$$

where in the antepenultimate line we have made use of Lemma 13.3.1. This completes the proof of (13.3.43) and hence the proof of Theorem 13.3.1 for $\operatorname{Re} z > 0$. But both sides of (13.3.4) are analytic for $|\arg z| < \pi$. Hence, by analytic continuation, the theorem is true for all complex z such that $|\arg z| < \pi$. □

Y. Lee [210] has also devised a proof of Entry 13.3.1.

13.4 Page 195

On page 195 in [269], Ramanujan defines

$$
\phi(x) := \psi(x) + \frac{1}{2x} - \gamma - \log x \qquad (13.4.1)
$$

and then concludes that

$$
\sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty} \phi(n\alpha) \right\} = \sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty} \phi(n\beta) \right\} \qquad (13.4.2)
$$

$$
= -\sqrt{\alpha} \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \left(\frac{1}{e^{x\alpha} - 1} - \frac{1}{x\alpha} \right) dx \qquad (13.4.3)
$$

$$
= -\frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t \log \alpha\right)}{1+t^2} dt. \qquad (13.4.4)
$$

First, in view of the asymptotic expansion (13.2.28) and the definition (13.4.1), the series in (13.4.2) do not converge. Second, the equality of the expressions in (13.4.3) and (13.4.4) does not hold. For equality to exist, the expression in (13.4.3) must be replaced by (see equation (22) of [257])

$$
-\int_0^{\infty} \left(\frac{1}{e^{x\sqrt{\beta}} - 1} - \frac{1}{x\sqrt{\beta}} \right) \left(\frac{1}{e^{x\sqrt{\alpha}} - 1} - \frac{1}{x\sqrt{\alpha}} \right) dx.
$$

13.5 Analogues of Entry 13.3.1

A. Dixit [108, 109] has established two beautiful analogues of Entry 13.3.1. Previously, a finite analogue of Theorem 13.5.1 was established by L. Carlitz [86].

Theorem 13.5.1. *Let $\zeta(z,a)$ denote the Hurwitz zeta function defined for $a>0$ and $\operatorname{Re} z>1$ by*

$$\zeta(z,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^z}.$$

If α and β are positive numbers such that $\alpha\beta=1$, then for $\operatorname{Re} z>2$ and $1<c<\operatorname{Re} z-1$,

$$\begin{aligned}
\alpha^{-z/2}\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\alpha}\right)&=\beta^{-z/2}\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\beta}\right)\\
&=\frac{\alpha^{z/2}}{2\pi i\Gamma(z)}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds\\
&=\frac{8(4\pi)^{(z-4)/2}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\\
&\times\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}dt,
\end{aligned}$$

where $\Xi(t)$ is defined in (13.3.1).

Theorem 13.5.2. *Let $0<\operatorname{Re} z<2$. Define $\varphi(z,x)$ by*

$$\varphi(z,x)=\zeta(z,x)-\frac{1}{2}x^{-z}+\frac{x^{1-z}}{1-z},$$

where $\zeta(z,x)$ denotes the Hurwitz zeta function. Then if α and β are any positive numbers such that $\alpha\beta=1$,

$$\begin{aligned}
&\alpha^{z/2}\left(\sum_{n=1}^{\infty}\varphi(z,n\alpha)-\frac{\zeta(z)}{2\alpha^z}-\frac{\zeta(z-1)}{\alpha(z-1)}\right)\\
&=\beta^{z/2}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta)-\frac{\zeta(z)}{2\beta^z}-\frac{\zeta(z-1)}{\beta(z-1)}\right)\\
&=\frac{8(4\pi)^{(z-4)/2}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\\
&\times\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt,
\end{aligned}\tag{13.5.1}$$

where $\Xi(t)$ is defined in (13.3.1).

If we let $z \to 1$ in (13.5.1), then we obtain Ramanujan's transformation (13.3.3). Thus Theorem 13.5.2 is a generalization of Entry 13.3.1. In [109], Dixit also obtained an analogue of Theorem 13.5.2 for $-3 < \operatorname{Re} z < -1$. Another generalization of the first identity in Entry 13.3.1 has been found by O. Oloa [237]. Another proof, employing a theorem on the double cotangent function, has been given by H. Tanaka [299].

13.6 Added Note: Pages 193, 194, 250

On pages 193 and 194 in [269], Ramanujan offers several Fourier and Laplace transforms, most of which are found in Entries 13.2.1 and 13.2.2. Since all of the results are standard in the theory of Fourier transforms, there is no need to repeat them here. On page 250 there appears some scratch work on Laplace transforms; no identities are recorded. The third integral on the page appears to be related to [255, Eq. (16)], [267, p. 56].

14

Integral Analogues of Theta Functions and Gauss Sums

14.1 Introduction

In this chapter we discuss a second partial manuscript of two pages [269, pp. 221–222] as well as a related page from the original lost notebook [269, p. 198]. As previously indicated, this manuscript does not belong to the "official" lost notebook of Ramanujan, but instead is among the eight partial manuscripts in G.N. Watson's handwriting that were found in the Oxford University library and that were published along with the lost notebook; the original version for these two pages is in the library at Trinity College, Cambridge. Pages 221 and 222 provide a list of theorems, with no discourse, on integrals that are found in Ramanujan's two papers [256, 258], [267, pp. 59–67] and [194–199]; see also [247]. Indeed, most of the theorems can be found in these two papers, especially [258]. Since Ramanujan did not give many details in these two papers, we shall provide proofs for each claim, whether it is found in these two papers or not.

The objective in the two papers cited above and in the two page fragment is the study of the functions

$$\phi_w(t) := \int_0^\infty \frac{\cos(\pi t x)}{\cosh(\pi x)} e^{-\pi w x^2}\,dx, \tag{14.1.1}$$

$$\psi_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\sinh(\pi x)} e^{-\pi w x^2}\,dx. \tag{14.1.2}$$

It is clear from the definitions (14.1.1) and (14.1.2) that, respectively,

$$\phi_w(t) = \phi_w(-t) \quad \text{and} \quad \psi_w(t) = -\psi_w(-t). \tag{14.1.3}$$

Page 198 of [269] is an isolated page that is actually part of the original lost notebook, and its contents are related to pages 221–222. On this page,

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Ramanujan records theorems, much in the spirit of those for $\phi_w(t)$ and $\psi_w(t)$, for the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\tanh(\pi x)} e^{-\pi w x^2} dx.$$

The theorems on page 198 are new and were first proved in a paper by Berndt and P. Xu [69].

The functions $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$ examined in this chapter and (for the former two functions) in [256, 258], [267, pp. 59–67, 202–207] can be regarded as continuous analogues of theta functions, because they each possess a transformation formula like that for the classical theta functions. For example, recall that the classical theta function

$$\theta_3(\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \operatorname{Im} \tau > 0,$$

satisfies the transformation formula [306, p. 22]

$$\theta_3(-1/\tau) = \sqrt{\tau/i}\, \theta_3(\tau). \tag{14.1.4}$$

On the other hand, because of the appearance of certain sums, which are reminiscent of Gauss sums, in the quasiperiodic relations, for example, in Entries 14.4.2 and 14.4.3, where the quasiperiods are $2i$ and $2w$, respectively, Ramanujan perhaps preferred the analogy with Gauss sums. Recall that the generalized Gauss sum $S(a,b,c)$, where a, b, and c are integers with $ac \neq 0$, is defined by

$$S(a,b,c) := \sum_{n=0}^{|c|-1} e^{\pi i (an^2+bn)/c}.$$

These sums satisfy a reciprocity theorem; namely, if $ac+b$ is even, then [54, p. 13]

$$S(a,b,c) = \sqrt{|c/a|} e^{\pi i \{\operatorname{sgn}(ac) - b^2/(ac)\}/4} S(-c,-b,a).$$

Note that on comparing the two sides of this identity, the roles of a and c are reversed. Moreover, $\sqrt{|c/a|}$ takes the place of $\sqrt{\tau}$ in (14.1.4) or $\sqrt{w}$ in the transformation formulas for $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$.

Because these functions possess quasiperiods $2i$ and $2w$, they can also be regarded as analogues of elliptic functions. For example, the Weierstrass σ-function is defined by

$$\sigma(z) := \sigma(z; \omega_1, \omega_2) := z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} - \frac{z^2}{2\omega^2}\right),$$

where $\omega = m\omega_1 + n\omega_2$, $-\infty < m, n < \infty$, and $\operatorname{Im} \omega_2/\omega_1 > 0$. Set $\omega_3 = \omega_1 + \omega_2$ and $\eta_j = \zeta(\omega_j/2)$, $j = 1,2,3$, where $\zeta(t)$ denotes the Weierstrass ζ-function. Then the Weierstrass σ-function obeys the quasiperiodic relations [88, p. 52]

$$\sigma(z+\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j/2)}, \quad j = 1, 2, 3.$$

Interesting analogues of the integrals studied by Ramanujan in [256] and [258] that involve Bessel functions have been derived by N.S. Koshliakov [192]. Those taking the qualifying examination in mathematics at Harvard University in fall 1998, day 2, were asked to evaluate a special case of $\phi_w(t)$.

14.2 Values of Useful Integrals

Throughout our proofs, we appeal to several integral evaluations, all of which can be found in the *Tables* of I.S. Gradshteyn and I.M. Ryzhik [126]. First [126, p. 515, formulas 3.898, nos. 1, 2], for $\operatorname{Re}\beta > 0$,

$$\int_0^\infty e^{-\beta x^2} \sin(ax)\sin(bx)dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-(a-b)^2/(4\beta)} - e^{-(a+b)^2/(4\beta)}\right\}, \tag{14.2.1}$$

$$\int_0^\infty e^{-\beta x^2} \cos(ax)\cos(bx)dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-(a-b)^2/(4\beta)} + e^{-(a+b)^2/(4\beta)}\right\}. \tag{14.2.2}$$

Second [126, p. 400, formula 3.546, no. 2], for $\operatorname{Re}\beta > 0$,

$$\int_0^\infty e^{-\beta x^2} \cosh(ax)dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}e^{a^2/(4\beta)}. \tag{14.2.3}$$

Third [126, p. 536, formula 3.981, no. 1], for $\operatorname{Re}\beta > 0$ and $a > 0$,

$$\int_0^\infty \frac{\sin(ax)}{\sinh(\beta x)}dx = \frac{\pi}{2\beta}\tanh\left(\frac{a\pi}{2\beta}\right). \tag{14.2.4}$$

Fourth [126, p. 552, formula 4.133, no. 1], for $\operatorname{Re}\gamma > 0$,

$$\int_0^\infty e^{-x^2/(4\gamma)} \sin(ax)\sinh(\beta x)dx = \sqrt{\pi\gamma}e^{\gamma(\beta^2-a^2)}\sin(2a\beta\gamma). \tag{14.2.5}$$

14.3 The Claims in the Manuscript

We now examine in order the claims made by Ramanujan on pages 221 and 222.

Entry 14.3.1 (p. 221). *For* $w > 0$,

$$\phi_w(t) = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)}\phi_{1/w}(it/w). \tag{14.3.1}$$

Proof. Using (13.2.21), inverting the order of integration, employing (14.2.2), and simplifying, we find that

$$\begin{aligned}\phi_w(t) &= 2\int_0^\infty \int_0^\infty \frac{\cos(2\pi xz)}{\cosh(\pi z)}\cos(\pi tx)e^{-\pi wx^2}dzdx\\ &= 2\int_0^\infty \frac{dz}{\cosh(\pi z)}\int_0^\infty \cos(2\pi xz)\cos(\pi tx)e^{-\pi wx^2}dx\\ &= 2\int_0^\infty \frac{1}{\cosh(\pi z)}\frac{1}{4}\sqrt{\frac{1}{w}}\left\{e^{-\frac{(2\pi z-\pi t)^2}{4\pi w}}+e^{-\frac{(2\pi z+\pi t)^2}{4\pi w}}\right\}\\ &= \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)}\int_0^\infty \frac{\cosh(\pi zt/w)}{\cosh(\pi z)}e^{-\pi z^2/w}dz, \qquad (14.3.2)\end{aligned}$$

which is equivalent to (14.3.1). □

A different proof of Entry 14.3.1 has been given by Y. Lee [210].

Entry 14.3.2 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)}\phi_w(t+w) = e^{\pi t^2/(4w)}\left(\frac{1}{2}+\psi_w(t)\right). \qquad (14.3.3)$$

Proof. First observe from (14.2.3) that

$$\int_0^\infty \cosh(\pi tx/w)e^{-\pi x^2/w}dx = \frac{1}{2}\sqrt{w}e^{\pi t^2/(4w)} \qquad (14.3.4)$$

and from (14.2.4) that

$$\int_0^\infty \frac{\sin(2\pi xz)}{\sinh(\pi z)}dz = \frac{1}{2}\tanh(\pi x). \qquad (14.3.5)$$

Thus, using (14.3.2), (14.3.4), and (14.3.5), we find that

$$\begin{aligned}&\phi_w(t+w)\\ &= \frac{1}{\sqrt{w}}e^{-\pi(t+w)^2/(4w)} \qquad (14.3.6)\\ &\quad\times\int_0^\infty \frac{\cosh(\pi tx/w)\cosh(\pi x)+\sinh(\pi tx/w)\sinh(\pi x)}{\cosh(\pi x)}e^{-\pi x^2/w}dx\\ &= \frac{1}{\sqrt{w}}e^{-\pi(t+w)^2/(4w)}\left\{\frac{1}{2}\sqrt{w}e^{\pi t^2/(4w)}\right.\\ &\qquad\left.+2\int_0^\infty\int_0^\infty \frac{\sin(2\pi xz)}{\sinh(\pi z)}\sinh(\pi tx/w)e^{-\pi x^2/w}dz\,dx\right\}.\end{aligned}$$

Now, by (14.2.5),

$$
\begin{aligned}
2\int_0^\infty \int_0^\infty & \frac{\sin(2\pi xz)}{\sinh(\pi z)} \sinh(\pi tx/w) e^{-\pi x^2/w} dz\, dx \\
&= 2\int_0^\infty \frac{dz}{\sinh(\pi z)} \int_0^\infty \sin(2\pi xz)\sinh(\pi tx/w)e^{-\pi x^2/w} dx \\
&= 2\int_0^\infty \frac{1}{\sinh(\pi z)} \frac{1}{2}\sqrt{w} e^{\pi t^2/(4w)} e^{-\pi z^2 w} \sin(\pi tz) dz \\
&= \sqrt{w} e^{\pi t^2/(4w)} \int_0^\infty \frac{\sin(\pi tz)}{\sinh(\pi z)} e^{-\pi z^2 w} dz \\
&= \sqrt{w} e^{\pi t^2/(4w)} \psi_w(t).
\end{aligned}
$$

If we use this last calculation in (14.3.6) and manipulate slightly, we complete the proof of (14.3.3). □

Entry 14.3.3 (p. 221). *We have*

$$
\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i\right)\right\}. \tag{14.3.7}
$$

Proof. Rewrite (14.3.3) as

$$
\frac{1}{2} + \psi_w(t) = e^{\pi t/2 + \pi w/4} \phi_w(t+w). \tag{14.3.8}
$$

Thus, using (14.3.8), (14.3.1), (14.1.3), and (14.3.3) with w replaced by $1/w$, we find that

$$
\begin{aligned}
\frac{1}{2} &+ \psi_w(t+i) = ie^{\pi t/2+\pi w/4}\phi_w(t+i+w) \\
&= ie^{\pi t/2+\pi w/4}\frac{1}{\sqrt{w}} e^{-\pi(t+i+w)^2/(4w)}\phi_{1/w}(i(t+i+w)/w) \\
&= \frac{i}{\sqrt{w}} e^{-\pi(t^2-1+2it+2iw)/(4w)}\phi_{1/w}\left(-\frac{it}{w} - i + \frac{1}{w}\right) \\
&= \frac{i}{\sqrt{w}} e^{-\pi(t^2-1+2it+2iw)/(4w)} e^{-\pi(-it/w-i)/2-\pi/(4w)} \left\{\frac{1}{2} + \psi_{1/w}\left(-\frac{it}{w} - i\right)\right\} \\
&= \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{\frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i\right)\right\},
\end{aligned}
$$

where in the last step we used (14.1.3). Hence, (14.3.7) has been established. □

Entry 14.3.4 (p. 221). *We have the evaluations*

$$
\phi_w(i) = \frac{1}{2\sqrt{w}}, \tag{14.3.9}
$$

$$
\psi_w(i) = \frac{i}{2\sqrt{w}}, \tag{14.3.10}
$$

$$\phi_w(w) = \frac{1}{2}e^{-\pi w/4}, \tag{14.3.11}$$

$$\frac{1}{2} - \psi_w(w) = e^{-\pi w/4}\phi_w(0). \tag{14.3.12}$$

Proof. From the definition (14.1.1),

$$\phi_w(i) = \int_0^\infty e^{-\pi w x^2}dx = \frac{1}{2\sqrt{w}},$$

and from the definition (14.1.2),

$$\psi_w(i) = i\int_0^\infty e^{-\pi w x^2}dx = \frac{i}{2\sqrt{w}}.$$

Next, by the functional equation (14.3.1),

$$\phi_w(w) = \frac{1}{\sqrt{w}}e^{-\pi w/4}\phi_{1/w}(i) = \frac{1}{2}e^{-\pi w/4},$$

upon the use of (14.3.9). Lastly, by (14.3.3) with $t=-w$ and by (14.1.3),

$$e^{\pi w/4}\left\{\frac{1}{2} + \psi_w(-w)\right\} = e^{\pi w/4}\left\{\frac{1}{2} - \psi_w(w)\right\} = \phi_w(0),$$

and so the final assertion (14.3.12) of our entry has been proved. □

Entry 14.3.5 (p. 221). *We have*

$$\phi_w(w\pm i) = \left(\frac{1}{2\sqrt{w}} \mp \frac{i}{2}\right)e^{-\pi w/4}, \tag{14.3.13}$$

$$\psi_w(w\pm i) = \frac{1}{2} \pm \frac{i}{2\sqrt{w}}e^{-\pi w/4}, \tag{14.3.14}$$

$$\phi_w\left(\frac{1}{2}w\right) + \psi_w\left(\frac{1}{2}w\right) = \frac{1}{2}. \tag{14.3.15}$$

Proof. Using (14.3.3), (14.1.3), and (14.3.10) and then simplifying, we find that

$$\phi_w(w\pm i) = e^{-\pi(\pm i+w)^2/(4w)-\pi/(4w)}\left\{\frac{1}{2} + \psi_w(\pm i)\right\} = e^{-\pi w/4}\left\{\mp\frac{i}{2} + \frac{1}{2\sqrt{w}}\right\},$$

which completes the proof of (14.3.13).

Appealing to (14.3.7) with $t=\pm w$ and using (14.1.3), we find that

$$\frac{1}{2} \pm \psi_w(w\pm i) = \frac{i}{\sqrt{w}}e^{-\pi w/4}\left\{\frac{1}{2} - \psi_{1/w}(\pm i + i)\right\}. \tag{14.3.16}$$

We need to distinguish two cases in (14.3.16). First,

$$\begin{aligned}\psi_{1/w}(2i) &= i\int_0^\infty \frac{\sinh(2\pi x)}{\sinh(\pi x)} e^{-\pi x^2/w}dx \\ &= 2i\int_0^\infty \cosh(\pi x)e^{-\pi x^2/w}dx = i\sqrt{w}e^{\pi w/4},\end{aligned} \tag{14.3.17}$$

by (14.2.3). Using the calculation from (14.3.17) in (14.3.16) and simplifying, we find that

$$\psi_w(w+i) = \frac{i}{2\sqrt{w}}e^{-\pi w/4} + \frac{1}{2},$$

as claimed in (14.3.14). Second, we observe that trivially $\psi_{1/w}(0) = 0$, and so in the second case, (14.3.16) reduces to

$$\frac{1}{2} - \psi_w(w-i) = \frac{i}{\sqrt{w}}e^{-\pi w/4}\frac{1}{2},$$

which immediately gives the other evaluation in (14.3.14). Third, return to (14.3.3) and set $t = -\frac{1}{2}w$ to deduce that

$$\frac{1}{2} - \psi_w\left(\frac{1}{2}w\right) = \phi_w\left(\frac{1}{2}w\right),$$

which is what we wanted to prove. □

Entry 14.3.6 (p. 221). *We have*

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)}, \tag{14.3.18}$$

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}. \tag{14.3.19}$$

Proof. Using the definition (14.1.1), elementary trigonometric identities, and (14.2.2), we find that

$$\phi_w(t+i) + \phi_w(t-i) = 2\int_0^\infty \cos(\pi t x)e^{-\pi w x^2}dx = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)},$$

as claimed in (14.3.18).

Next, employing (14.1.2), further elementary trigonometric identities, and (14.2.2) once again, we see that

$$\psi_w(t+i) - \psi_w(t-i) = 2i\int_0^\infty \cos(\pi t x)e^{-\pi w x^2}dx = \frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)},$$

which is (14.3.19). □

Entry 14.3.7 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)}\phi_w(t+w)+e^{\pi(t-w)^2/(4w)}\phi_w(t-w)=e^{\pi t^2/(4w)}. \qquad (14.3.20)$$

Proof. Employing the identity (14.3.3) and the oddness of $\psi(t)$ noted in (14.1.3), we readily find that

$$\begin{aligned}&e^{\pi(t+w)^2/(4w)}\phi_w(t+w)+e^{\pi(t-w)^2/(4w)}\phi_w(t-w)\\&\quad=e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}+e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(-t)\right\}=e^{\pi t^2/(4w)},\end{aligned}$$

which is identical to (14.3.20). □

Entry 14.3.8 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+w)\right\}=e^{\pi(t-w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t-w)\right\}. \qquad (14.3.21)$$

Proof. Appealing to (14.3.3) and then using (14.1.3), we find that

$$\begin{aligned}e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}&=e^{\pi(t+w)^2/(4w)}\phi_w(t+w)\\&=e^{\pi(t+w)^2/(4w)}\phi_w(-t-w). \qquad (14.3.22)\end{aligned}$$

Replacing t by $-t-w$ above and using (14.1.3), we arrive at

$$e^{\pi(t+w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+w)\right\}=e^{\pi t^2/(4w)}\phi_w(t).$$

Replacing t by $t-w$ in (14.3.22) and using (14.1.3), we deduce that

$$e^{\pi(t-w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t-w)\right\}=e^{\pi t^2/(4w)}\phi_w(t).$$

The identity (14.3.21) is now an immediate consequence of the last two identities. □

Entry 14.3.9 (p. 221). *If n is any positive integer, then*

$$\phi_w(t)+(-1)^{n+1}\phi_w(t+2ni)=\frac{1}{\sqrt{w}}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t+(2k+1)i)^2/(4w)}. \qquad (14.3.23)$$

Proof. We employ (14.3.18) with t successively replaced by $t+i, t+3i, \dots, t+(2n-1)i$ to deduce the array

$$\begin{aligned}
\phi_w(t+2i)+\phi_w(t) &= \frac{1}{\sqrt{w}}e^{-\pi(t+i)^2/(4w)},\\
\phi_w(t+4i)+\phi_w(t+2i) &= \frac{1}{\sqrt{w}}e^{-\pi(t+3i)^2/(4w)},\\
&\ \vdots\\
\phi_w(t+2ni)+\phi_w(t+(2n-2)i) &= \frac{1}{\sqrt{w}}e^{-\pi(t+(2n-1)i)^2/(4w)}.
\end{aligned}$$

Alternately adding and subtracting the identities above, we immediately deduce (14.3.23). □

Entry 14.3.10 (p. 221). *If n is any positive integer,*

$$\psi_w(t)-\psi_w(t+2ni) = -\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1}e^{-\pi(t+(2k+1)i)^2/(4w)}. \tag{14.3.24}$$

Proof. We employ (14.3.19) with t successively replaced by $t+i, t+3i, \dots, t+(2n-1)i$, and so record the identities

$$\begin{aligned}
\psi_w(t+2i)-\psi_w(t) &= \frac{i}{\sqrt{w}}e^{-\pi(t+i)^2/(4w)},\\
\psi_w(t+4i)-\psi_w(t+2i) &= \frac{i}{\sqrt{w}}e^{-\pi(t+3i)^2/(4w)},\\
&\ \vdots\\
\psi_w(t+2ni)-\psi_w(t+(2n-2)i) &= \frac{i}{\sqrt{w}}e^{-\pi(t+(2n-1)i)^2/(4w)}.
\end{aligned}$$

Adding the identities above, we deduce (14.3.24) forthwith. □

Entry 14.3.11 (p. 221). *For any positive integer n,*

$$\begin{aligned}
e^{\pi t^2/(4w)}\phi_w(t)+(-1)^{n+1}e^{\pi(t+2nw)^2/(4w)}\phi_w(t+2nw)&\\
=\sum_{k=0}^{n-1}(-1)^k e^{\pi(t+(2k+1)w)^2/(4w)}.&
\end{aligned} \tag{14.3.25}$$

Proof. We return to (14.3.20) and successively replace t by $t+w, t+3w, \dots, t+(2n-1)w$ to deduce the n equations

$$\begin{aligned}
&e^{\pi(t+2w)^2/(4w)}\phi_w(t+2w)+e^{\pi t^2/(4w)}\phi_w(t) = e^{\pi(t+w)^2/(4w)},\\
&e^{\pi(t+4w)^2/(4w)}\phi_w(t+4w)+e^{\pi(t+2w)^2/(4w)}\phi_w(t+2w) = e^{\pi(t+3w)^2/(4w)},\\
&\qquad\vdots\\
&e^{\pi(t+2nw)^2/(4w)}\phi_w(t+2nw)+e^{\pi(t+(2n-2)w)^2/(4w)}\phi_w(t+(2n-2)w)\\
&\qquad = e^{\pi(t+(2n-1)w)^2/(4w)}.
\end{aligned}$$

If we now alternately add and subtract the identities above, we readily deduce (14.3.25). □

Entry 14.3.12 (p. 221). *For any positive integer n,*

$$e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}+(-1)^{n+1}e^{\pi(t+2nw)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+2nw)\right\} = \sum_{k=1}^{n}(-1)^{k-1}e^{\pi(t+2kw)^2/(4w)}. \tag{14.3.26}$$

Proof. We apply (14.3.21) with t successively replaced by $t+w, t+3w, \dots, t+(2n-1)w$ in order to derive the set of equations

$$e^{\pi(t+2w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+2w)\right\} = e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\},$$
$$e^{\pi(t+4w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+4w)\right\} = e^{\pi(t+2w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+2w)\right\},$$
$$\vdots$$
$$e^{\pi(t+2nw)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+2nw)\right\} = e^{\pi(t+(2n-2)w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+(2n-2)w)\right\}.$$

We now alternately add and subtract these identities to achieve (14.3.26). □

Entry 14.3.13 (p. 222). *Let m and n denote any positive integers and set $s=t+2mw\pm 2ni$. Then*

$$\phi_w(s)+(-1)^{(m+1)(n+1)}e^{-\frac{1}{2}\pi m(s+t)}\phi_w(t) = e^{-\pi s^2/(4w)}\sum_{k=0}^{m-1}(-1)^k e^{\pi(s-(2k+1)w)^2/(4w)} + \frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t\pm(2k+1)i)^2/(4w)}. \tag{14.3.27}$$

Proof. We first observe that an analogue to Entry 14.3.9 can be obtained by beginning the proof with the relation

$$\phi_w(t)+\phi_w(t-2i) = \frac{1}{\sqrt{w}}e^{-\pi(t-i)^2/(4w)}.$$

Proceeding as before, we can then deduce that

$$\phi_w(t)+(-1)^{n+1}\phi_w(t-2ni) = \frac{1}{\sqrt{w}}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t-(2k+1)i)^2/(4w)}. \tag{14.3.28}$$

We apply Entry 14.3.11 with n replaced by m, where m is a positive integer, and then with t replaced by $t \pm 2ni$, where n is a positive integer. After rearranging and using the definition of s, we find that

$$\begin{aligned}
\phi_w(s) &+ (-1)^{m+1} e^{-\pi m s + \pi m^2 w} \phi_w(t \pm 2ni) \\
&= \phi_w(s) + (-1)^{m+1} e^{-\pi s^2/(4w) + \pi (t \pm 2ni)^2/(4w)} \phi_w(t \pm 2ni) \\
&= (-1)^{m+1} e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi (t \pm 2ni + (2j-1)w)^2/(4w)} \\
&= e^{-\pi s^2/(4w)} \sum_{r=0}^{m-1} (-1)^r e^{\pi (s-(2r+1)w)^2/(4w)}, \qquad (14.3.29)
\end{aligned}$$

where we changed the index of summation by setting $j = m - r$.

Next, we apply Entry 14.3.9 and its analogue (14.3.28) to see that

$$(-1)^{n+1} \phi_w(t \pm 2ni) + \phi_w(t) = \frac{1}{\sqrt{w}} \sum_{k=0}^{n-1} (-1)^k e^{-\pi (t \pm (2k+1)i)^2/(4w)}.$$

Upon multiplying both sides by

$$(-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)},$$

we find that

$$\begin{aligned}
(-1)^{n+1+(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)} \phi_w(t \pm 2ni) + (-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)} \phi_w(t) \\
= \frac{(-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)}}{\sqrt{w}} \sum_{k=0}^{n-1} (-1)^k e^{-\pi (t \pm (2k+1)i)^2/(4w)}. \qquad (14.3.30)
\end{aligned}$$

We now add (14.3.29) and (14.3.30) and observe, with the aid of the definition of s, that the coefficient of $\phi_w(t \pm 2ni)$ is equal to

$$(-1)^{m+1} e^{-\pi m s + \pi m^2 w} + (-1)^{m(n+1)} e^{-\frac{1}{2}\pi m (s+t)} = 0. \qquad (14.3.31)$$

We thus immediately obtain (14.3.27) to complete the proof. □

Entry 14.3.14 (p. 222). *Let m and n denote positive integers. Then, if $s = 2mw \pm 2ni$,*

$$\begin{aligned}
\frac{1}{2} - \psi_w(s) &+ (-1)^{mn+m+1} e^{-\frac{1}{2}\pi m (s+t)} \left\{ \frac{1}{2} - \psi_w(t) \right\} \\
&= e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi (s - 2jw)^2/(4w)} \\
&\quad \pm \frac{(-1)^{mn+m+1} i}{\sqrt{w}} e^{-\frac{1}{2}\pi m (s+t)} \sum_{j=0}^{n-1} e^{-\pi (t \pm (2j+1)i)^2/(4w)}. \qquad (14.3.32)
\end{aligned}$$

Proof. If we examine the proof of Entry 14.3.10, we see that we can obtain an analogue of (14.3.24), just as we previously obtained (14.3.28), except that now the right-hand side is multiplied by -1. Hence,

$$\psi_w(t) - \psi_w(t \pm 2ni) = \mp \frac{i}{\sqrt{w}} \sum_{j=0}^{n-1} e^{-\pi(t+(2j+1)i)^2/(4w)}. \tag{14.3.33}$$

We apply Entry 14.3.12 with n replaced by m, and then with t replaced by $t \pm 2ni$. Next multiply both sides by $(-1)^{m+1}e^{-\pi s^2/(4w)}$. Setting also $j = m + 1 - r$ below, we find that

$$\begin{aligned}
&\frac{1}{2} + \psi_w(s) + (-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t \pm 2ni) \right\} \\
&\qquad = (-1)^{m+1} e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi(t\pm 2ni+2jw)^2/(4w)} \\
&\qquad = -e^{-\pi s^2/(4w)} \sum_{r=1}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)} \\
&\qquad = 1 - e^{-\pi s^2/(4w)} \sum_{r=2}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)}.
\end{aligned}$$

Rearranging, we deduce that

$$\begin{aligned}
&\frac{1}{2} - \psi_w(s) - (-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t \pm 2ni) \right\} \\
&\qquad = e^{-\pi s^2/(4w)} \sum_{r=2}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)}.
\end{aligned} \tag{14.3.34}$$

Observe that with the definition of s,

$$(-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} = (-1)^{m+1} e^{-\pi m s + \pi m^2 w}. \tag{14.3.35}$$

We also observe that if $r = m + 1$, the corresponding expression (including $e^{-\pi s^2/(4w)}$) on the right-hand side of (14.3.34) is also equal to the right-hand side of (14.3.35). We add this expression to both sides of (14.3.34) and replace r by $r + 1$, so that we can rewrite (14.3.34) in the form

$$\begin{aligned}
&\frac{1}{2} - \psi_w(s) + (-1)^{m+1} e^{-\pi m s + \pi m^2 w} \left\{ \frac{1}{2} - \psi_w(t \pm 2ni) \right\} \\
&\qquad = e^{-\pi s^2/(4w)} \sum_{r=1}^{m} (-1)^{r-1} e^{\pi(s-2rw)^2/(4w)}.
\end{aligned} \tag{14.3.36}$$

Multiply both sides of (14.3.33) by

$$-(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}$$

to deduce that

$$(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}\left(\left\{\frac{1}{2}-\psi_w(t)\right\}-\left\{\frac{1}{2}-\psi_w(t\pm 2ni)\right\}\right)$$
$$=\pm\frac{(-1)^{mn+m+1}i}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\sum_{j=0}^{n-1}e^{-\pi(t+(2j+1)i)^2/(4w)}. \tag{14.3.37}$$

We now add (14.3.36) and (14.3.37). Observe that, with the definition of s, the coefficient of $\frac{1}{2}-\psi_w(t\pm 2ni)$ equals

$$(-1)^{m+1}e^{-\pi ms+\pi m^2 w}-(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}=0,$$

by the same calculation as in (14.3.31). We thus immediately deduce (14.3.32) to complete the proof. □

Entry 14.3.15 (p. 222). *Let $t = mw \pm ni$, where m and n are positive integers. If m is odd and n is odd, or if m is even and n is odd, or if m is odd and n is even, then*

$$\phi_w(t)=\frac{1}{2}e^{-\pi t^2/(4w)}\sum_{j=0}^{m-1}(-1)^j e^{\pi(t-(2j+1)w)^2/(4w)}$$
$$+\frac{1}{2\sqrt{w}}\sum_{j=0}^{n-1}(-1)^j e^{\pi(t\mp(2j+1)i)^2/(4w)}. \tag{14.3.38}$$

Proof. In Entry 14.3.13, replace t by $-t$ and then set $s=t$. Thus, t has the form stated in the present entry. In all three cases, (14.3.27) readily reduces to (14.3.38). □

Entry 14.3.16 (p. 222). *Let $t = mw \pm ni$, where m and n are positive integers. If m is odd and n is odd, or if m is even and n is odd, or if m is even and n is even, then*

$$\psi_w(t)=-\frac{1}{2}e^{-\pi t^2/(4w)}\sum_{j=1}^{m}(-1)^{j-1}e^{\pi(t-2jw)^2/(4w)}$$
$$\pm\frac{i}{2\sqrt{w}}\sum_{j=0}^{n-1}e^{\pi(t\mp(2j+1)i)^2/(4w)}. \tag{14.3.39}$$

Proof. The proof is similar to the previous proof. In Entry 14.3.14, replace t by $-t$ and then set $s=t$. In all three cases, (14.3.32) simplifies to (14.3.39). □

We quote Ramanujan for the last claim on page 222 of [269]. If $t = mw \pm ni$, then

$$\phi_w(t) = e^{-\frac{1}{4}\pi m^2 w} \left\{ \frac{1}{2} \left(\frac{1}{\sqrt{w}} + e^{\mp \frac{1}{2}\pi i m n} \right) \sin \tfrac{1}{2}\pi m ?. \right. \tag{14.3.40}$$

Evidently, the presence of the question mark indicates that Ramanujan was unsure of his claim and that further terms (possibly unknown to Ramanujan) were needed to complete the identity. As (14.3.40) is presently stated, it is not true in general. For example, if $m = 2$ and $n = 1$, (14.3.40) is false.

14.4 Page 198

Page 198 in the lost notebook is devoted to properties of the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\tanh(\pi x)} e^{-\pi w x^2} dx. \tag{14.4.1}$$

The formulas claimed by Ramanujan on page 198 are difficult to read, partly because the original page was perhaps a thin, colored piece of paper, for example, a piece of parchment paper, that was difficult to photocopy.

It is clear from the definition (14.4.1) that

$$F_w(t) = -F_w(-t). \tag{14.4.2}$$

Entry 14.4.1 (p. 198). *We have*

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w). \tag{14.4.3}$$

Proof. Write

$$\begin{aligned} F_w(t) &= \int_0^\infty \frac{\sin(\pi t x)\cosh(\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \int_0^\infty \frac{\sin(\pi t x)\cos(i\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{1}{2} \int_0^\infty \frac{\sin(t+i)\pi x + \sin(t-i)\pi x}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{1}{2} \left\{ \psi_w(t+i) + \psi_w(t-i) \right\}, \end{aligned} \tag{14.4.4}$$

by (14.1.2). Recall from (14.3.7) that

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i \right) \right\}. \tag{14.4.5}$$

Since $\psi(t)$ is odd, we find from (14.4.5) that

$$-\frac{1}{2}+\psi_w(t-i) = -\frac{1}{2}-\psi_w(-t+i) = -\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(-\frac{it}{w}+i\right)\right\}. \tag{14.4.6}$$

Hence, from (14.4.4)–(14.4.6),

$$\begin{aligned}
F_w(t) &= \frac{1}{2}\left\{\frac{1}{2}+\psi_w(t+i)-\frac{1}{2}+\psi_w(t-i)\right\}\\
&= \frac{1}{2}\left(\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(\frac{it}{w}+i\right)\right\}\right.\\
&\qquad\left.-\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(-\frac{it}{w}+i\right)\right\}\right)\\
&= \frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(-\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(-\frac{it}{w}+i\right)\right)\\
&= -\frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(\frac{it}{w}-i\right)\right)\\
&= -\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}F_{1/w}(it/w),
\end{aligned}$$

by (14.4.4), and this completes the proof. □

Entry 14.4.2 (p. 198). *If n is any positive integer, then*

$$F_w(t)-F_w(t+2ni) = -\frac{i}{\sqrt{w}}{\sum_{j=0}^{n}}' e^{-\pi(t+2ji)^2/(4w)}, \tag{14.4.7}$$

where the prime $'$ on the summation sign indicates that the terms with $j=0,n$ are to be multiplied by $\frac{1}{2}$.

Proof. Recall from (14.4.4) that

$$F_w(t) = \frac{1}{2}\{\psi_w(t+i)+\psi_w(t-i)\}, \tag{14.4.8}$$

and so

$$\begin{aligned}
&F_w(t)-F_w(t+2ni)\\
&\quad= \frac{1}{2}\{\psi_w(t+i)-\psi_w(t+(2n+1)i)\}+\frac{1}{2}\{\psi_w(t-i)-\psi_w(t+(2n-1)i)\}.
\end{aligned}$$

Applying Entry 14.3.10 on the right side above, we see that

$$\begin{aligned}
&F_w(t) - F_w(t+2ni) \\
&\quad = \frac{1}{2}\left\{-\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1} e^{-\pi(t+(2k+2)i)^2/(4w)} - \frac{i}{\sqrt{w}}\sum_{k=0}^{n-1} e^{-\pi(t+2ki)^2/(4w)}\right\} \\
&\quad = -\frac{i}{\sqrt{w}}\sum_{j=0}^{n}{}' e^{-\pi(t+2ji)^2/(4w)}.
\end{aligned}$$

This concludes the proof. □

Entry 14.4.3 (p. 198). *If* n *is a positive integer, then*

$$F_w(t) - e^{\pi n(t+nw)} F_w(t+2nw) = -e^{-\pi t^2/(4w)} \sum_{j=0}^{n}{}' e^{\pi(t+2jw)^2/(4w)}, \tag{14.4.9}$$

where the prime on the summation sign has the same meaning as in Entry 14.4.2.

Proof. Replacing t by $t+i$ and $t-i$ in Entry 14.3.8, we deduce, respectively, that

$$\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}\psi_w(t+i+w) + e^{\pi(t+i-w)^2/(4w)}\psi_w(t+i-w) \\
&\qquad = \frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)} - e^{\pi(t+i-w)^2/(4w)}\right)
\end{aligned} \tag{14.4.10}$$

and

$$\begin{aligned}
&e^{\pi(t-i+w)^2/(4w)}\psi_w(t-i+w) + e^{\pi(t-i-w)^2/(4w)}\psi_w(t-i-w) \\
&\qquad = \frac{1}{2}\left(e^{\pi(t-i+w)^2/(4w)} - e^{\pi(t-i-w)^2/(4w)}\right).
\end{aligned} \tag{14.4.11}$$

Now observe that $e^{4\pi i(t+w)/(4w)} = e^{4\pi i(t-w)/(4w)}$. We multiply $e^{\pi(t-i+w)^2/(4w)}$ in its two appearances in (14.4.11) by $e^{4\pi i(t+w)/(4w)}$, and we multiply $e^{\pi(t-i-w)^2/(4w)}$ in its two appearances in (14.4.11) by $e^{4\pi i(t-w)/(4w)}$. Thus, (14.4.11) can be recast in the form

$$\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}\psi_w(t-i+w) + e^{\pi(t+i-w)^2/(4w)}\psi_w(t-i-w) \\
&\qquad = \frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)} - e^{\pi(t+i-w)^2/(4w)}\right).
\end{aligned} \tag{14.4.12}$$

Using (14.4.8), (14.4.10), and (14.4.12), we find that

$$
\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}F_w(t+w)+e^{\pi(t+i-w)^2/(4w)}F_w(t-w)\\
&\quad=\frac{1}{2}\left\{e^{\pi(t+i+w)^2/(4w)}\psi_w(t+i+w)+e^{\pi(t+i+w)^2/(4w)}\psi_w(t-i+w)\right.\\
&\qquad\left.+e^{\pi(t+i-w)^2/(4w)}\psi_w(t+i-w)+e^{\pi(t+i-w)^2/(4w)}\psi_w(t-i-w)\right\}\\
&\quad=\frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)}-e^{\pi(t+i-w)^2/(4w)}\right). \qquad (14.4.13)
\end{aligned}
$$

We now apply (14.4.13) with t successively replaced by $t+w$, $t+3w$, $\dots$, $t+(2n-1)w$ to deduce the n equations

$$
\begin{aligned}
&e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)+e^{\pi(t+i)^2/(4w)}F_w(t)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+2w)^2/(4w)}-e^{\pi(t+i)^2/(4w)}\right),\\
&e^{\pi(t+i+4w)^2/(4w)}F_w(t+4w)+e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+4w)^2/(4w)}-e^{\pi(t+i+2w)^2/(4w)}\right),\\
&\qquad\qquad\qquad\vdots\\
&e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)+e^{\pi(t+i+(2n-2)w)^2/(4w)}F_w(t+(2n-2)w)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+2nw)^2/(4w)}-e^{\pi(t+i+(2n-2)w)^2/(4w)}\right).
\end{aligned}
$$

Alternately adding and subtracting the identities above, we conclude that

$$
\begin{aligned}
&e^{\pi(t+i)^2/(4w)}F_w(t)+(-1)^{n+1}e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)\\
&\qquad=\sum_{j=0}^{n}{}'(-1)^{j+1}e^{\pi(t+i+2jw)^2/(4w)},
\end{aligned}
$$

that is to say,

$$
F_w(t)-e^{\pi n(t+nw)}F_w(t+2nw)=-e^{-\pi t^2/(4w)}\sum_{j=0}^{n}{}'e^{\pi(t+2jw)^2/(4w)},
$$

which completes our proof. □

Entry 14.4.4 (p. 198). *Let $s=t+2\eta_1 mw+2\eta_2 ni$, where $\eta_1^2=\eta_2^2=1$, and where m and n are positive integers. Then*

$$
\begin{aligned}
F_w(s)+(-1)^{mn-1}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}F_w(t)&=\eta_1 e^{-\pi s^2/(4w)}\sum_{j=0}^{m}{}'e^{\pi(s-2j\eta_1 w)^2/(4w)}\\
&+\eta_2(-1)^{mn}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}\sum_{j=0}^{n}{}'e^{-\pi(t+2\eta_2 ji)^2/(4w)}, \quad (14.4.14)
\end{aligned}
$$

where the primes on the summation signs have the same meaning as in the two previous entries.

Proof. If we examine the proof of Entry 14.4.3, we see that we can similarly obtain an expression for $F_w(t) - e^{-\pi n(t-nw)}F_w(t-2nw)$, but with the right-hand side multiplied by -1 and the exponents j in the summands being replaced by $-j$. Thus, we shall apply Entry 14.4.3 and its just described analogue with n replaced by m and t replaced by $t+2\eta_2 ni$. Note that the right-hand side will be multiplied by η_1, and so we obtain

$$F_w(t+2\eta_2 ni) - e^{\pi\eta_1 m(t+2\eta_2 ni+\eta_1 mw)}F_w(t+2\eta_2 ni+2\eta_1 mw) = -\eta_1 e^{-\pi(t+2\eta_2 ni)^2/(4w)}\sideset{}{'}\sum_{j=0}^{m} e^{\pi(t+2\eta_2 ni+2\eta_1 jw)^2/(4w)}.$$

Using the definition of s, we can reformulate the foregoing equality as

$$\begin{aligned} F_w(t+2\eta_2 ni) - e^{\pi\eta_1 m(s-\eta_1 mw)}F_w(s) &= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)}\sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 mw+2\eta_1 jw)^2/(4w)} \\ &= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)}\sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 jw)^2/(4w)}. \end{aligned} \tag{14.4.15}$$

If we examine the proof of Entry 14.4.2, we see that we can obtain an analogue for $F_w(t)-F_w(t-2ni)$ with the right-hand side now being multiplied by -1 and with the summand exponents j replaced by $-j$. Then if we apply Entry 14.4.2 and its analogue that we just described above to $F_w(t+2\eta_2 ni)$, we must multiply the right-hand side by η_2. Hence, using (14.4.7), its analogue, and (14.4.15), we find that

$$F_w(t) - e^{\pi\eta_1 m(s-\eta_1 mw)}F_w(s) = -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)}\sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2\eta_1 jw)^2/(4w)} - \eta_2\frac{i}{\sqrt{w}}\sideset{}{'}\sum_{j=0}^{n} e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

Upon multiplying both sides above by $e^{-\pi\eta_1 m(s-\eta_1 mw)}$ and simplifying, we find that

$$\begin{aligned} F_w(s) + (-1)^{mn-1}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}F_w(t) &= -\eta_1 e^{-\pi s^2/(4w)}\sideset{}{'}\sum_{j=0}^{m} e^{\pi(s-2j\eta_1 w)^2/(4w)} \\ &+ \eta_2(-1)^{mn}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}\sideset{}{'}\sum_{j=0}^{n} e^{\pi(t+2\eta_2 ji)^2/(4w)}, \end{aligned}$$

where we used the fact that

$$(-1)^{mn}e^{-\frac{1}{2}\pi\eta_1 m(s+t)} = e^{-\pi\eta_1 m(s-\eta_1 mw)}.$$

This completes our proof. □

14.5 Examples

If we set $s = t$ in Entry 14.4.4, it follows that $w = -(\eta_2 ni)/(\eta_1 m)$. If we further suppose that both m and n are odd, then (14.4.14) reduces to the identity

$$(1 + e^{-\pi\eta_1 mt})F_w(t) = \eta_1 e^{-\pi t^2/(4w)} \sum_{j=0}^{m}{}' e^{\pi(t-2j\eta_1 w)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} e^{-\pi\eta_1 mt} \sum_{j=0}^{n}{}' e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

In the identity above, first let $\eta_1 = 1, \eta_2 = -1$ and multiply both sides by e^{mt}. Second, let $\eta_1 = -1, \eta_2 = 1$ and multiply both sides by e^{-mt}. Replace t by $2t/\pi$ in each identity. We then respectively obtain the two identities

$$\begin{aligned} 2\cosh(mt) &\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m} i} dx \\ &= \frac{1}{2}e^{mt} + e^{(m-2)t + \frac{\pi n}{m} i} + e^{(m-4)t + \frac{4\pi n}{m} i} + \cdots + \frac{1}{2}e^{-mt+\pi mni} \\ &\quad + \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} e^{-mt + \left(\frac{mt^2}{\pi n} + \frac{\pi}{4}\right)i} + e^{\left(\frac{2}{n}-1\right)mt + \left[\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n} + \frac{\pi}{4}\right]i} \right. \\ &\quad \left. + \cdots + \frac{1}{2} e^{mt + \left[\left(\frac{t^2}{\pi^2} - n^2\right)\frac{\pi m}{n} + \frac{\pi}{4}\right]i} \right\} \end{aligned} \tag{14.5.1}$$

and

$$\begin{aligned} 2\cosh(mt) &\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)} e^{-\frac{\pi n x^2}{m} i} dx \\ &= -\frac{1}{2}e^{-mt} - e^{(2-m)t + \frac{\pi n}{m} i} - e^{(4-m)t + \frac{4\pi n}{m} i} + \cdots - \frac{1}{2}e^{mt+\pi mni} \\ &\quad - \sqrt{\frac{m}{n}} \left\{ \frac{1}{2} e^{mt + \left(\frac{mt^2}{\pi n} + \frac{\pi}{4}\right)i} + e^{\left(1-\frac{2}{n}\right)mt + \left[\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n} + \frac{\pi}{4}\right]i} \right. \\ &\quad \left. + \cdots + \frac{1}{2} e^{-mt + \left[\left(\frac{t^2}{\pi^2} - n^2\right)\frac{\pi m}{n} + \frac{\pi}{4}\right]i} \right\}. \end{aligned} \tag{14.5.2}$$

Next add (14.5.1) and (14.5.2), divide both sides by 2, and equate the real and imaginary parts on both sides to obtain the two identities

$$\begin{aligned}
&2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\cos\frac{\pi n x^2}{m}dx \\
&\quad= \frac{1}{2}\sinh\{mt\} + \sinh\{(m-2)t\}\cos\frac{\pi n}{m} + \sinh\{(m-4)t\}\cos\frac{4\pi n}{m} \\
&\qquad + \cdots + \frac{1}{2}\sinh\{-mt\}\cos(\pi m n) \\
&\qquad + \sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\cos\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right) + \sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\right. \\
&\qquad\qquad \times\cos\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right) \\
&\qquad\qquad \left. + \cdots + \frac{1}{2}\sinh\{mt\}\cos\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\} \qquad (14.5.3)
\end{aligned}$$

and

$$\begin{aligned}
&-2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin\frac{\pi n x^2}{m}dx \\
&\quad= \sinh\{(m-2)t\}\sin\frac{\pi n}{m} + \sinh\{(m-4)t\}\sin\frac{4\pi n}{m} \\
&\qquad + \cdots + \frac{1}{2}\sinh\{-mt\}\sin(\pi m n) \\
&\qquad + \sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\sin\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right) + \sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\right. \\
&\qquad\qquad \times\sin\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right) \\
&\qquad\qquad \left. + \cdots + \frac{1}{2}\sinh\{mt\}\sin\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\}. \qquad (14.5.4)
\end{aligned}$$

Using (14.5.3) and (14.5.4), we can evaluate several definite integrals. For example, if we set $m = n = 1$ in (14.5.3) and (14.5.4), we find that, respectively,

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\cos(\pi x^2)dx = \frac{\sinh t}{2\cosh t}\left(1-\cos\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right)\right)$$

and

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin(\pi x^2)dx = \frac{\sinh t}{2\cosh t}\sin\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right).$$

These evaluations can be found in [126, p. 542, formulas 3.991, nos. 1, 2], respectively. No further cases of (14.5.3) and (14.5.4) can be found in [126].

14.6 One Further Integral

There is one further integral, namely,

$$G_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\coth(\pi x)} e^{-\pi w x^2} dx,$$

that can be placed in the theory of $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$. Note that

$$\begin{aligned} G_w(t) &= \int_0^\infty \frac{\sin(\pi t x)\sinh(\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= -i \int_0^\infty \frac{\sin(\pi t x)\sin(i\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{i}{2} \int_0^\infty \frac{\cos\{\pi x(t+i)\} - \cos\{\pi x(t-i)\}}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{i}{2}\{\phi_w(t+i) - \phi_w(t-i)\}, \qquad (14.6.1) \end{aligned}$$

by (14.1.1). The formula (14.6.1) should be compared with (14.3.18).

15

Functional Equations for Products of Mellin Transforms

15.1 Introduction

Pages 223–227 in [269] form a third manuscript originally written by Ramanujan but in the handwriting of G.N. Watson. These five pages are devoted to finding solutions to a certain functional equation for products of Mellin transforms. At the beginning of the manuscript, sufficient details are provided, but in the latter portions of the manuscript fewer details are given, especially for a lengthy series of examples illustrating one of Ramanujan's theorems. As did Ramanujan, we shall proceed formally. Hypotheses from the theory of Mellin transforms can be added to ensure validity of the processes. The manuscript is divided into three sections. We follow Ramanujan's development throughout, although, as we shall see, the organization in this rough manuscript is not optimal. Because the manuscript comprises continuous discourse, we have refrained from setting aside claims and using the designation "Entry" in this chapter.

15.2 Statement of the Main Problem

Let

$$X_1(s) := \int_0^\infty x^{s-1}\chi_1(x)dx \qquad \text{and} \qquad X_2(s) := \int_0^\infty x^{s-1}\chi_2(x)dx. \quad (15.2.1)$$

The functions χ_1 and χ_2 are to be chosen so that the functional equation

$$X_1(s)X_2(1-s) = \lambda^2 \quad (15.2.2)$$

holds, where λ is independent of s.

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Ramanujan then claims that the functional equations

$$\begin{cases} \int_0^\infty \phi(x)\chi_1(nx)dx = \lambda\psi(n), \\ \int_0^\infty \psi(x)\chi_2(nx)dx = \lambda\phi(n) \end{cases} \tag{15.2.3}$$

imply each other. This claim is best established after further theory is developed.

From (15.2.1),

$$X_1(\tfrac{1}{2}) = \int_0^\infty x^{-1/2}\chi_1(x)dx \qquad \text{and} \qquad X_2(\tfrac{1}{2}) = \int_0^\infty x^{-1/2}\chi_2(x)dx. \tag{15.2.4}$$

Setting $x = y^2$ in (15.2.4), we find that

$$X_1(\tfrac{1}{2}) = 2\int_0^\infty \chi_1(y^2)dy \qquad \text{and} \qquad X_2(\tfrac{1}{2}) = 2\int_0^\infty \chi_2(y^2)dy. \tag{15.2.5}$$

Since by (15.2.2), $X_1(\frac{1}{2})X_2(\frac{1}{2}) = \lambda^2$, we conclude from (15.2.5) that

$$\int_0^\infty \chi_1(y^2)dy \int_0^\infty \chi_2(y^2)dy = \left(\frac{1}{2}\lambda\right)^2. \tag{15.2.6}$$

Suppose now that $\chi_1(x)$ and $\chi_2(x)$ are replaced by $a_1\chi_1(cx)$ and $a_2\chi_2(cx)$, respectively, where a_1, a_2, and c are constants. Then, if $cx = t$ and $j = 1, 2$,

$$X_j(s) = \int_0^\infty x^{s-1}a_j\chi_j(cx)dx = a_jc^{-s}\int_0^\infty t^{s-1}\chi_j(t)dt.$$

It then follows from (15.2.2) that

$$X_1(s)X_2(1-s) = a_1a_2c^{-s}c^{-1+s}\lambda^2 = \frac{a_1a_2}{c}\lambda^2. \tag{15.2.7}$$

Hence, the aforementioned substitutions imply that in (15.2.2), λ must be replaced by $\sqrt{a_1a_2/c}\,\lambda$, or if we set $\lambda^* = \sqrt{a_1a_2/c}\,\lambda$, then, from (15.2.2),

$$X_1(s)X_2(1-s) = \lambda^{*2}.$$

Now let

$$Z_1(s) := \int_0^\infty x^{s-1}\phi(x)dx \qquad \text{and} \qquad Z_2(s) := \int_0^\infty x^{s-1}\psi(x)dx. \tag{15.2.8}$$

Then, by (15.2.3),

$$\begin{aligned} Z_1(s) &= \frac{1}{\lambda}\int_0^\infty x^{s-1}dx \int_0^\infty \psi(y)\chi_2(xy)dy \\ &= \frac{1}{\lambda}\int_0^\infty \psi(y)dy \int_0^\infty x^{s-1}\chi_2(xy)dx \\ &= \frac{1}{\lambda}\int_0^\infty y^{-s}\psi(y)dy \int_0^\infty t^{s-1}\chi_2(t)dt \\ &= \frac{1}{\lambda}Z_2(1-s)X_2(s). \end{aligned} \tag{15.2.9}$$

Hence, by (15.2.2),

$$\frac{Z_1(s)}{Z_2(1-s)} = \frac{X_2(s)}{\lambda} = \frac{\lambda}{X_1(1-s)}. \tag{15.2.10}$$

We assume throughout the sequel that a given pair of functions $F(s)$ and $f(x)$ are related by Mellin transforms, that is to say,

$$F(s) = \int_0^\infty x^{s-1}f(x)dx \qquad \text{and} \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}ds. \tag{15.2.11}$$

We now justify Ramanujan's claim that the two equations in (15.2.3) imply each other. In fact, we prove that Eqs. (15.2.2) and (15.2.10) imply both equations in (15.2.3), so that in this sense they are equivalent. We prove the first equation in (15.2.3); the proof of the second is analogous. To that end, by (15.2.8), (15.2.11), (15.2.10), (15.2.2), (15.2.1), (15.2.11), and (15.2.8),

$$\begin{aligned} \psi(n) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_2(s)n^{-s}ds \\ &= \frac{\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{Z_1(1-s)}{X_2(1-s)}n^{-s}ds \\ &= \frac{1}{2\pi i\lambda}\int_{c-i\infty}^{c+i\infty} Z_1(1-s)X_1(s)n^{-s}ds \\ &= \frac{1}{2\pi i\lambda}\int_{c-i\infty}^{c+i\infty} Z_1(1-s)n^{-s}ds \int_0^\infty x^{s-1}\chi_1(x)dx \\ &= \frac{1}{\lambda}\int_0^\infty \chi_1(nt)dt \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_1(1-s)t^{s-1}ds \\ &= \frac{1}{\lambda}\int_0^\infty \chi_1(nt)\phi(t)dt, \end{aligned}$$

which is what we wanted to prove.

Ramanujan next completes Sect. 15.2 with several examples.

Example 15.2.1. If

$$\begin{cases} \chi_1(x) = \chi_2(x) = \cos x, \\ \qquad\qquad \text{or} \\ \chi_1(x) = \chi_2(x) = \sin x, \end{cases} \tag{15.2.12}$$

then

$$\lambda^2 = \frac{1}{2}\pi. \tag{15.2.13}$$

To establish the first part of Example 15.2.1, recall that for $\operatorname{Re} s > 0$ [126, p. 458, formula 3.761, no. 9],

$$\int_0^\infty x^{s-1} \cos x \, dx = \Gamma(s) \cos\left(\frac{\pi s}{2}\right).$$

Hence,

$$\begin{aligned} X_1(s)X_2(1-s) &= \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)\cos\left(\frac{\pi(1-s)}{2}\right) \\ &= \frac{\pi}{\sin(\pi s)}\cos\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi s}{2}\right) = \frac{\pi}{2}, \end{aligned}$$

where we employed the reflection formula for the gamma function.

For the second part of Example 15.2.1, recall that for $\operatorname{Re} s > 0$ [126, p. 458, formula 3.761, no. 4],

$$\int_0^\infty x^{s-1} \sin x \, dx = \Gamma(s) \sin\left(\frac{\pi s}{2}\right).$$

Hence,

$$\begin{aligned} X_1(s)X_2(1-s) &= \Gamma(s)\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sin\left(\frac{\pi(1-s)}{2}\right) \\ &= \frac{\pi}{\sin(\pi s)}\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right) = \frac{\pi}{2}. \end{aligned}$$

Example 15.2.2. If

$$\chi_1(x) = \chi_2(x) = J_\nu(x)\sqrt{x}, \tag{15.2.14}$$

where J_ν denotes the ordinary Bessel function of order ν and $\operatorname{Re}\nu > -1$, then

$$\lambda = 1. \tag{15.2.15}$$

To establish (15.2.15), recall that for $-\operatorname{Re}\nu - \frac{1}{2} < \operatorname{Re} s < 1$ [126, p. 707, formula 6.561, no. 14],

$$\int_0^\infty x^{s-1}\sqrt{x} J_\nu(x) dx = 2^{s-1/2} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{4})}.$$

Hence,

$$X_1(s)X_2(1-s) = 2^{s-1/2}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}2^{-s+1/2}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})} = 1.$$

Example 15.2.3. If

$$\chi_1(x) = \chi_2(x) = \frac{x^\nu}{1+x^2},$$

where ν is an integer, then

$$\lambda = \frac{\pi}{2}. \tag{15.2.16}$$

In order to prove (15.2.16), recall that [126, p. 341, formula 3.241, no. 3]

$$\text{PV}\int_0^\infty \frac{x^{\nu+s-1}}{1-x^2}dx = \frac{\pi}{2}\cot\frac{(\nu+s)\pi}{2}, \quad \text{Re}(\nu+s)>0,$$

where PV denotes the principal value of the integral. Hence,

$$\begin{aligned} X_1(s)X_2(1-s) &= \frac{\pi^2}{4}\cot\frac{(\nu+s)\pi}{2}\cot\frac{(\nu+1-s)\pi}{2} \\ &= \frac{\pi^2}{4}\frac{\cot\frac{\nu\pi}{2}\cot\frac{s\pi}{2}-1}{\cot\frac{\nu\pi}{2}+\cot\frac{s\pi}{2}}\cdot\frac{-\cot\frac{(\nu+1)\pi}{2}\cot\frac{s\pi}{2}-1}{\cot\frac{(\nu+1)\pi}{2}-\cot\frac{s\pi}{2}} \\ &= \frac{\pi^2}{4}, \end{aligned}$$

where it is helpful to consider the cases ν even and odd separately.

Ramanujan concludes page 223 by giving examples in which "ϕ and ψ are the same function." The first two examples provide self-reciprocal Fourier cosine and sine transforms, respectively.

Example 15.2.4. We have

$$\int_0^\infty e^{-x^2}\cos(2nx)dx = \frac{\sqrt{\pi}}{2}e^{-n^2}, \tag{15.2.17}$$

$$\int_0^\infty xe^{-x^2}\sin(2nx)dx = \frac{n\sqrt{\pi}}{2}e^{-n^2}. \tag{15.2.18}$$

The identities (15.2.17) and (15.2.18) are given in the *Tables* [126, p. 515, formula 3.896, no. 4; p. 529, formula 3.952, no. 1], respectively.

The next example is misprinted in [269]; Ramanujan (or Watson) wrote $J_\nu(nx)$ instead of $J_\nu(2nx)$. The example gives a self-reciprocal transform with respect to the kernel $\sqrt{nx}\,J_\nu(2nx)$.

Example 15.2.5. We have

$$\int_0^\infty x^{\nu+1/2}e^{-x^2}\sqrt{nx}J_\nu(2nx)dx = \frac{1}{2}n^{\nu+1/2}e^{-n^2}. \tag{15.2.19}$$

The integral evaluation (15.2.19) can be found in [126, p. 738, formula 6.631, no. 4].

Example 15.2.6. We have

$$\text{PV}\int_0^\infty \frac{1}{1+x^2}\frac{dx}{1-n^2x^2} = \frac{\pi}{2(1+n^2)}. \tag{15.2.20}$$

The evaluation (15.2.20) is located in [126, p. 348, formula 3.264, no. 1].

15.3 The Construction of χ_1 and χ_2

We have used Ramanujan's title for this section; he did not give a title for the first section. The first half of page 224 is clear and well written, with the example toward the end covered previously in Example 15.2.1. We therefore quote Ramanujan.

> If χ_1 is given it is theoretically (though not always) possible to find χ_2 with the help of (15.2.2) and (15.2.11). In this procedure χ_1 and χ_2 are generally neither the same function nor similar functions, nor both of them capable of finite expression at the same time. It is also extremely improbable that we can select functions like $\cos x$ or $\sin x$. We shall now proceed in a different way. It is always possible to choose a number of functions $X_1(s)$ and $X_2(s)$ which are either the same function or similar functions so that $X_1(s)X_2(1-s)$ is an absolute constant. Then by (15.2.11) we have
>
> $$\begin{cases} \chi_1(x) &= \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} x^{-s}X_1(s)ds, \\ \chi_2(x) &= \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} x^{-s}X_2(s)ds. \end{cases} \tag{15.3.1}$$
>
> For instance suppose that
>
> $$X_1(s) = X_2(s) = \Gamma(s)\cos\tfrac{1}{2}\pi s \quad \text{or} \quad \Gamma(s)\sin\tfrac{1}{2}\pi s \tag{15.3.2}$$
>
> so that $X_1(s)X_2(1-s)$ is a constant. Then from (15.3.1) we deduce that $\chi_1(x) = \chi_2(x) = \cos x$ or $\sin x$ by using the well-known formula
>
> $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)ds = e^{-x}. \tag{15.3.3}$$

The identity (15.3.3) is not tagged by Ramanujan. However, because the tag (15.3.3) is missing and because Ramanujan later refers to the identity as (15.3.3), we have inserted a tag.

We quote Ramanujan once more, but with one serious misprint corrected.

> Suppose now that the $2g$ integers, say
>
> $$0, 1, 2, \ldots, 2g-1$$
>
> are divided into any group of g integers, say
>
> $$\begin{aligned} &a_1, a_2, \ldots, a_g \\ &b_1, b_2, \ldots, b_g \end{aligned}$$
>
> and that
>
> $$\begin{cases} X_1(s) &= \Gamma(s) \prod_{r=1}^{g} \sin \dfrac{\pi(s+a_r)}{2g}, \\ X_2(s) &= \Gamma(s) \prod_{r=1}^{g} \sin \dfrac{\pi(s-b_r+2g-1)}{2g} \end{cases} \tag{15.3.4}$$
>
> so that $X_1(s)X_2(1-s)$ is a constant. Then we can easily find $\chi_1(x)$ and $\chi_2(x)$ with the help of (15.3.1) and (15.3.3).

It is interesting that many years later, in 1950, A.P. Guinand [134] rediscovered the same general set of examples as given by Ramanujan in (15.3.4). Guinand worked out all the specific cases for $g = 3$.

Ramanujan next asserts that the number of possible choices of $a_1, a_2, \ldots, a_g$ and $b_1, b_2, \ldots, b_g$ is

$$\frac{(2g)!}{(g!)^2}. \tag{15.3.5}$$

Ramanujan then defines these solutions in χ_1 and χ_2 to belong *to class g*. However, these solutions "include the number of solutions belonging to class δ, where δ is a divisor of g. Hence eliminating all these extraneous solutions we find the number of ways belonging to class g is"

$$\omega_g = \sum_{\delta} \mu(\delta) \frac{(2g/\delta)!}{\{(g/\delta)!\}^2}, \tag{15.3.6}$$

where the sum is over all divisors δ of g, and where μ denotes the Möbius function. The truth of (15.3.6) is easily seen by a straightforward application of the inclusion–exclusion principle. Ramanujan then provides a table of values of ω_g, $1 \le g \le 10$:

g	ω_g	g	ω_g
1	$\frac{2!}{(1!)^2} = 2$	6	$\frac{12!}{(6!)^2} - \frac{6!}{(3!)^2} - \frac{4!}{(2!)^2} + \frac{2!}{(1!)^2} = 900$
2	$\frac{4!}{(2!)^2} - \frac{2!}{(1!)^2} = 4$	7	$\frac{14!}{(7!)^2} - \frac{2!}{(1!)^2} = 3,430$
3	$\frac{6!}{(3!)^2} - \frac{2!}{(1!)^2} = 18$	8	$\frac{16!}{(8!)^2} - \frac{8!}{(4!)^2} = 12,800$
4	$\frac{8!}{(4!)^2} - \frac{4!}{(2!)^2} = 64$	9	$\frac{18!}{(9!)^2} - \frac{6!}{(3!)^2} = 48,600$
5	$\frac{10!}{(5!)^2} - \frac{2!}{(1!)^2} = 250$	10	$\frac{20!}{(10!)^2} - \frac{10!}{(5!)^2} - \frac{4!}{(2!)^2} + \frac{2!}{(1!)^2} = 184,500$

Table 15.1. Calculation of ω_g

At this point, we demonstrate that with the choices made in (15.3.4), $X_1(s)X_2(1-s)$ is a constant. To that end, by the reflection formula for $\Gamma(s)$,

$$\begin{aligned} X_1(s)X_2(1-s) &= \Gamma(s)\Gamma(1-s)\prod_{r=1}^{g}\sin\frac{\pi(s+a_r)}{2g}\sin\frac{\pi(-s-b_r+2g)}{2g} \\ &= \frac{\pi}{\sin(\pi s)}\prod_{r=1}^{g}\sin\frac{\pi(s+a_r)}{2g}\sin\frac{\pi(s+b_r)}{2g} \\ &= \frac{\pi}{\sin(\pi s)}\prod_{r=0}^{2g-1}\sin\frac{\pi(s+r)}{2g} \\ &= \frac{\pi}{\sin(\pi s)}\frac{\sin(\pi s)}{2^{2g-1}} = \frac{\pi}{2^{2g-1}}, \end{aligned} \tag{15.3.7}$$

where we used a well-known value for the foregoing product of sines [126, p. 41, formula 1.392, no. 1].

After giving Table 15.1, Ramanujan sketchily gives the following table of inverse Mellin transforms for χ_1 and χ_2.

g	Inverse Mellin transforms
1	Trigonometric functions only
2	Trigonometric functions and e^{-x}
3	Trigonometric functions and $e^{-x\sqrt{3}}$
4	Trigonometric functions and e^{-x} and $e^{-x\sqrt{2}}$
5	Trigonometric functions and e^{-2x} and $e^{-x(1+\sqrt{5})}$
6	Trigonometric functions and e^{-x}, e^{-2x}, and $e^{-x\sqrt{3}}$

Table 15.2. Inverse Mellin transforms

We now justify Ramanujan's claims in Table 15.2. For $g = 1$, see the discourse after (15.3.2). These inverse Mellin transforms

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin\frac{\pi s}{2}x^{-s}ds = \sin x, \quad -1 < \operatorname{Re} s < 1, \tag{15.3.8}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\cos\frac{\pi s}{2}x^{-s}ds = \cos x, \quad 0 < \operatorname{Re} s < 1, \tag{15.3.9}$$

are well known and can be found, for instance, in the *Tables* of A. Erdélyi [115, p. 348, Eqs. (6) and (7)].

Second, let $g = 2$. Take $a_1 = 0$ and $a_2 = 1$. Then

$$X_1(s) = \Gamma(s)\sin\frac{\pi s}{4}\sin\frac{\pi(s+1)}{4} = \frac{1}{2}\Gamma(s)\left\{\cos\frac{\pi}{4} - \cos\left(\frac{\pi s}{2}+\frac{\pi}{4}\right)\right\}.$$

A similar formula holds for $X_2(s)$. Hence, by (15.3.3), (15.3.8), and (15.3.9), the inverse Mellin transforms for $X_1(s)$ and $X_2(s)$ will involve trigonometric functions and e^{-x}, as claimed by Ramanujan.

Third, set $g = 3$. In general,

$$\begin{aligned} X_1(s) &= \Gamma(s)\sin\frac{\pi(s+a_1)}{6}\sin\frac{\pi(s+a_2)}{6}\sin\frac{\pi(s+a_3)}{6} \\ &= \frac{1}{2}\Gamma(s)\left\{\cos\frac{\pi(a_2-a_1)}{6} - \cos\left(\frac{\pi s}{3}+\frac{\pi(a_1+a_2)}{6}\right)\right\}\sin\frac{\pi(s+a_3)}{6} \\ &= \frac{1}{4}\Gamma(s)\left\{\sin\left(\frac{\pi s}{6}+\frac{\pi c_1}{6}\right) + \sin\left(\frac{\pi s}{6}+\frac{\pi c_2}{6}\right)\right. \\ &\qquad\qquad \left. -\sin\left(\frac{\pi s}{2}+\frac{\pi c_3}{6}\right) + \sin\left(\frac{\pi s}{6}+\frac{\pi c_4}{6}\right)\right\}, \end{aligned}$$

for certain constants c_1, c_2, c_3, and c_4. A similar formula holds for $X_2(s)$. Now for $-\frac{1}{2}\pi < \operatorname{Re}\alpha < \frac{1}{2}\pi$ [115, p. 348, Eqs. (9) and (11)],

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin(\alpha s)x^{-s}ds = e^{-x\cos\alpha}\sin(x\sin\alpha), \quad -1 < \operatorname{Re} s, \tag{15.3.10}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\cos(\alpha s)x^{-s}ds = e^{-x\cos\alpha}\cos(x\sin\alpha), \quad 0 < \operatorname{Re} s. \tag{15.3.11}$$

Hence, using (15.3.8)–(15.3.11), we see that the inverse Mellin transforms of $X_1(s)$ and $X_2(s)$ can be expressed in terms of trigonometric functions and $e^{-x\sqrt{3}/2}$. Thus, after the replacement of x by $2x$, we see that Ramanujan's claim is correct.

Fourth, let $g = 4$. Then

$$\begin{aligned}X_1(s) &= \Gamma(s)\sin\frac{\pi(s+a_1)}{8}\sin\frac{\pi(s+a_2)}{8}\sin\frac{\pi(s+a_3)}{8}\sin\frac{\pi(s+a_4)}{8}\\ &= \frac{1}{4}\Gamma(s)\left\{\cos\frac{\pi(a_1-a_2)}{8}-\cos\left(\frac{\pi s}{4}+\frac{\pi(a_1+a_2)}{8}\right)\right\}\\ &\quad\times\left\{\cos\frac{\pi(a_3-a_4)}{8}-\cos\left(\frac{\pi s}{4}+\frac{\pi(a_3+a_4)}{8}\right)\right\}.\end{aligned}$$

It should now be clear that after applying further trigonometric identities and appealing to (15.3.3) and (15.3.8)–(15.3.11), we shall find that the inverse Mellin transform of $X_1(s)$ involves trigonometric functions, e^{-x}, and $e^{-x/\sqrt{2}}$. The argument for $X_2(s)$ is similar.

Fifth, set $g = 5$. Then

$$\begin{aligned}X_1(s) &= \Gamma(s)\sin\frac{\pi(s+a_1)}{10}\sin\frac{\pi(s+a_2)}{10}\sin\frac{\pi(s+a_3)}{10}\\ &\quad\times\sin\frac{\pi(s+a_4)}{10}\sin\frac{\pi(s+a_5)}{10}\\ &= \frac{1}{4}\Gamma(s)\left\{\cos\frac{\pi(a_1-a_2)}{10}-\cos\left(\frac{\pi s}{5}+\frac{\pi(a_1+a_2)}{10}\right)\right\}\\ &\quad\times\left\{\cos\frac{\pi(a_3-a_4)}{10}-\cos\left(\frac{\pi s}{5}+\frac{\pi(a_3+a_4)}{10}\right)\right\}\sin\frac{\pi(s+a_5)}{10}.\end{aligned}$$

After further applications of elementary trigonometric identities and the use of (15.3.8)–(15.3.11), we see that the inverse Mellin transforms of $X_1(s)$ will involve trigonometric functions, e^{-x}, $\exp(-x\sin\frac{\pi}{10}) = \exp(-x(-1+\sqrt{5})/4)$, and $\exp(-x\sin\frac{3\pi}{10}) = \exp(-x(1+\sqrt{5})/4)$. Thus, after replacing x by $2x$, we see that Ramanujan's claim is justified.

Finally, we consider the case $g = 6$. Then

$$\begin{aligned}X_1(s) &= \Gamma(s)\sin\frac{\pi(s+a_1)}{12}\sin\frac{\pi(s+a_2)}{12}\sin\frac{\pi(s+a_3)}{12}\\ &\quad\times\sin\frac{\pi(s+a_4)}{12}\sin\frac{\pi(s+a_5)}{12}\sin\frac{\pi(s+a_6)}{12}\\ &= \frac{1}{8}\Gamma(s)\left\{\cos\frac{\pi(a_1-a_2)}{12}-\cos\left(\frac{\pi s}{6}+\frac{\pi(a_1+a_2)}{12}\right)\right\}\\ &\quad\times\left\{\cos\frac{\pi(a_3-a_4)}{12}-\cos\left(\frac{\pi s}{6}+\frac{\pi(a_3+a_4)}{12}\right)\right\}\\ &\quad\times\left\{\cos\frac{\pi(a_5-a_6)}{12}-\cos\left(\frac{\pi s}{6}+\frac{\pi(a_5+a_6)}{12}\right)\right\}.\end{aligned}$$

Of course, we need to apply further trigonometric identities before calculating the inverse Mellin transform of $X_1(s)$. Using (15.3.3) and (15.3.8)–(15.3.11), we find that we obtain trigonometric functions, e^{-x}, $\exp(-x\cos\frac{\pi}{6})$, and $\exp(-x\cos\frac{\pi}{3})$. Thus, Ramanujan's claim again follows, although he has replaced x by $2x$ throughout.

15.4 The Case in Which $\chi_1(x) = \chi_2(x)$, $\phi(x) = \psi(x)$

The title of this section is that given by Ramanujan. In the first part, Ramanujan, in essence, repeats a portion of Sect. 15.2, but with the added restrictions given in the title of the section. Since the details are similar to those previously given, there is no need to elaborate on them here, and so we quote Ramanujan.

Let

$$X(s) = \int_0^\infty x^{s-1}\chi(x)dx,$$

and let the function $X(s)$ satisfy the relation

$$X(s)X(1-s) = \lambda^2, \tag{15.4.1}$$

where λ is independent of s. Then the two equations

$$\begin{cases} \displaystyle\int_0^\infty \phi(x)\chi(nx)dx = \lambda\psi(n), \\ \displaystyle\int_0^\infty \psi(x)\chi(nx)dx = \lambda\phi(n) \end{cases} \tag{15.4.2}$$

imply each other. It follows from (15.2.6) that

$$\frac{1}{2}\lambda = \pm\int_0^\infty \chi(x^2)dx. \tag{15.4.3}$$

We have already discussed about the construction of $\chi(x)$. We shall now consider the interesting case in which $\phi(x) = \psi(x)$. This is really two cases one in which λ has the positive sign and the other in which λ has the negative sign. It follows from (15.2.10) that if

$$Z(s) = \int_0^\infty x^{s-1}\phi(x)dx$$

and

$$\frac{Z(s)}{Z(1-s)} = \frac{X(s)}{\lambda} = \frac{\lambda}{X(1-s)}, \tag{15.4.4}$$

then

$$\int_0^\infty \phi(x)\chi(nx)dx = \lambda\phi(n). \tag{15.4.5}$$

Thus $\phi(x)$ is a positive reciprocal function of itself or a negative reciprocal function of itself according as λ is taken with positive or negative sign from (15.4.3). Suppose now that $T(s)$ is a function such that $T(s) = T(1-s)$ (for instance Riemann's $\xi(s)$ or some such function) and also that a solution of $Z(s)$ is found from (15.4.4). Then we can replace $Z(s)$ by $Z(s)T(s)$ and get a new ϕ function satisfying (15.4.5) for every $T(s)$ we choose. The following methods will show that the construction of ϕ is much easier than that of χ in the preceding section.

The next three sentences are mysterious, because the introduction of the function $f(x)$ is spurious. At any rate, we quote what Ramanujan and Watson recorded.

Let $f(x)$ be an arbitrary function. Then if

$$\phi(n) = f(n) + \frac{1}{\lambda}\int_0^\infty f(x)\chi(nx)dx \tag{15.4.6}$$

it follows from (15.4.2) that

$$\int_0^\infty \phi(x)\chi(nx)dx = \lambda\phi(n).$$

Here λ may have any of the two values in (15.4.3).

Let $f(x)$ be any function such that

$$f\left(\frac{1}{x}\right) = xf(x), \tag{15.4.7}$$

and let

$$\int_0^\infty \phi(x)\chi(nx)dx = \lambda\phi(n). \tag{15.4.8}$$

Then

$$\int_0^\infty F(x)\chi(nx)dx = \lambda F(n), \tag{15.4.9}$$

where

$$\int_\epsilon^{1/\epsilon} f(z)\phi(xz)dz = F(x), \tag{15.4.10}$$

where ϵ is any small positive number including zero. Ramanujan then says, “This is very easy to prove.” Indeed, using (15.4.10), setting $t = xz$, employing (15.4.8), setting $z = 1/u$, invoking (15.4.7), and lastly using (15.4.10) again, we find that

$$\begin{aligned}
\int_0^\infty F(x)\chi(nx)dx &= \int_0^\infty \int_\epsilon^{1/\epsilon} f(z)\phi(xz)\chi(nx)dzdx \\
&= \int_\epsilon^{1/\epsilon} f(z)\frac{dz}{z}\int_0^\infty \phi(t)\chi(nt/z)dt \\
&= \lambda\int_\epsilon^{1/\epsilon} \frac{f(z)}{z}\phi(n/z)dz \\
&= \lambda\int_\epsilon^{1/\epsilon} \frac{f(1/u)}{u}\phi(nu)du \\
&= \lambda\int_\epsilon^{1/\epsilon} f(u)\phi(nu)du \\
&= \lambda F(n),
\end{aligned}$$

and so the proof of (15.4.9) is complete.

As an illustration, let

$$f(x) := \frac{x^{\frac{1}{2}(ab-1)}}{(1+x^a)^b}e^{-u(x^v+x^{-v})}, \tag{15.4.11}$$

where a and b are arbitrary real numbers and the real part of u is positive. A straightforward calculation shows that $f(x)$ satisfies (15.4.7). Thus, (15.4.9) holds, where $F(x)$ is given by (15.4.10) and $\phi(x)$ satisfies (15.4.8). Ramanujan then gives three special cases. If $a = b = 1$, $\epsilon = 0$, and $u = 0$, then

$$F(n) = \int_0^\infty \frac{\phi(nx)}{1+x}dx.$$

If $a = 2$, $b = \frac{1}{2}$, $\epsilon = 0$, and $u = 0$, then

$$F(n) = \int_0^\infty \frac{\phi(nx)}{\sqrt{1+x^2}}dx.$$

For the third example, first replace v by $\frac{1}{2}v$ in (15.4.11). Now let a be arbitrary and $b = 0$. Then if $x = t^2$,

$$F(n) = \int_0^\infty x^{-1/2}e^{-u(x^{v/2}+x^{-v/2})}\phi(nx)dx = 2\int_0^\infty e^{-u(t^v+t^{-v})}\phi(nt^2)dt.$$

(In the Ramanujan–Watson manuscript, the factor 2 on the right-hand side above is missing.)

For Ramanujan's last example, return to Example 15.2.5, which we rewrite in the form

$$\int_0^\infty x^{\nu+1/2}e^{-x^2}\sqrt{2nx}J_\nu(2nx)dx = \frac{1}{\sqrt{2}}n^{\nu+1/2}e^{-n^2}.$$

Hence,

$$\chi_1(x) = \chi_2(x) = \sqrt{2x}J_\nu(2x), \qquad \lambda = 2^{-1/2},$$

and

$$\phi(x) = \psi(x) = \frac{1}{\sqrt{2}}x^{\nu+1/2}e^{-x^2}.$$

Now in (15.4.11), set $b = 0$ and $v = 2$. Then (15.4.10), with $f(x)$ as just stipulated and $\phi(x)$ as above, becomes

$$\begin{aligned} F(t) &= \int_\epsilon^{1/\epsilon} x^{-1/2}e^{-u(x^2+x^{-2})}\frac{1}{\sqrt{2}}(tx)^{\nu+1/2}e^{-t^2x^2}dx \\ &= \frac{t^{\nu+1/2}}{\sqrt{2}}\int_\epsilon^{1/\epsilon} x^\nu e^{-(u+t^2)x^2-u/x^2}dx. \end{aligned}$$

Thus, from (15.4.9), with $F(t)$ as given above,

$$\int_0^\infty F(x)\sqrt{2nx}J_\nu(2nx)dx = \frac{1}{\sqrt{2}}F(n).$$

We have corrected several misprints in the example above.

15.5 Examples

In the remainder of his rough draft of a potential paper, Ramanujan works out the values of $\chi_1(x)$ and $\chi_2(x)$ for $g = 1, 2, 3$ and all available possibilities for $a_1, a_2, \ldots, a_g$.

$g = 1$**,** $\omega_1 = 2$

First, take $a_1 = 0$. Then in (15.3.4) let

$$X_1(s) = X_2(s) = \Gamma(s)\sin\frac{\pi s}{2}.$$

By (15.3.7),

$$X_1(s)X_2(1-s) = \frac{\pi}{2}.$$

Then, by (15.3.1) and (15.3.8),

$$\chi_1(x) = \chi_2(x) = \sin x.$$

Second, take $a_1 = 1$. By (15.3.4),

$$X_1(s) = X_2(s) = \Gamma(s)\sin\frac{\pi(s+1)}{2}.$$

By (15.3.7),

$$X_1(s)X_2(1-s) = \frac{\pi}{2}.$$

Then, by (15.3.1) and (15.3.9),

$$\chi_1(x) = \chi_2(x) = \cos x.$$

$$g = 2, \ \omega_2 = 4$$

By (15.3.7), for all the examples for $g = 2$,

$$X_1(s)X_2(1-s) = \frac{\pi}{8}. \tag{15.5.1}$$

First, let $a_1 = 0$ and $a_2 = 1$. Then

$$X_1(s) := \Gamma(s) \sin\frac{\pi s}{4} \sin\frac{\pi(s+1)}{4}, \tag{15.5.2}$$

$$X_2(s) := \Gamma(s) \sin\frac{\pi(s+1)}{4} \sin\frac{\pi s}{4}. \tag{15.5.3}$$

Then, by (15.3.3), (15.3.9), and (15.3.8), for $j = 1, 2$,

$$\begin{aligned}
\chi_j(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) \sin\frac{\pi s}{4} \sin\frac{\pi(s+1)}{4} x^{-s} ds \\
&= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) \left\{\cos\frac{\pi}{4} - \cos\left(\frac{\pi s}{2} + \frac{\pi}{4}\right)\right\} x^{-s} ds \\
&= \frac{1}{4\sqrt{2}\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) \left\{1 - \cos\frac{\pi s}{2} + \sin\frac{\pi s}{2}\right\} x^{-s} ds \\
&= \frac{1}{2\sqrt{2}} \left\{e^{-x} - \cos x + \sin x\right\}.
\end{aligned}$$

These calculations are in agreement with Ramanujan, who gives

$$\lambda = \sqrt{\pi} \qquad \text{and} \qquad \chi_1(x) = \chi_2(x) = e^{-x} - \cos x + \sin x.$$

Thus, Ramanujan has multiplied each of (15.5.2) and (15.5.3) by $2\sqrt{2}$. Then, by (15.2.7), in place of (15.5.1), we would deduce that $X_1(s)X_2(1-s) = (2\sqrt{2})^2\pi/8 = \pi$.

Second, let $a_1 = 1$ and $a_2 = 2$. Then

$$X_1(s) := \Gamma(s) \sin\frac{\pi(s+1)}{4} \sin\frac{\pi(s+2)}{4},$$

$$X_2(s) := \Gamma(s) \sin\frac{\pi(s+3)}{4} \sin\frac{\pi s}{4}.$$

Foregoing the calculations, which are similar to those above, we find that

$$\begin{aligned}\chi_1(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin\frac{\pi(s+1)}{4}\sin\frac{\pi(s+2)}{4}x^{-s}ds \\ &= \frac{1}{2\sqrt{2}}\left\{\cos x + \sin x + e^{-x}\right\}\end{aligned}$$

and

$$\begin{aligned}\chi_2(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin\frac{\pi(s+3)}{4}\sin\frac{\pi s}{4}x^{-s}ds \\ &= \frac{1}{2\sqrt{2}}\left\{\cos x + \sin x - e^{-x}\right\}.\end{aligned}$$

Except for the fact that Ramanujan normalized $X_1(s)$ and $X_2(s)$ by multiplying each by $2\sqrt{2}$, our calculations agree with those of Ramanujan.

Third, set $a_1 = 0$ and $a_2 = 3$. Note that we will obtain the same representations for χ_1 and χ_2 that we did in the previous example, but with the roles of χ_1 and χ_2 reversed. In this set of examples with $g = 2$ and in the next set with $g = 3$, Ramanujan does not provide all the solutions in χ_1 and χ_2. However, it is to be understood that when χ_1 and χ_2 are *different*, then another entry, with their roles inverted, is an (absent) entry in the list as well.

Fourth, set $a_1 = 2$ and $a_2 = 3$. In this instance,

$$\begin{aligned}X_1(s) &:= \Gamma(s)\sin\frac{\pi(s+2)}{4}\sin\frac{\pi(s+3)}{4}, \\ X_2(s) &:= \Gamma(s)\sin\frac{\pi(s+3)}{4}\sin\frac{\pi(s+2)}{4}.\end{aligned}$$

With calculations not unlike those above, we find that for $j = 1, 2$,

$$\begin{aligned}\chi_j(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin\frac{\pi(s+2)}{4}\sin\frac{\pi(s+3)}{4}x^{-s}ds \\ &= \frac{1}{2\sqrt{2}}\left\{\cos x - \sin x + e^{-x}\right\},\end{aligned}$$

which, except for the factor $2^{-3/2}$, agrees with Ramanujan's claim.

$$g = 3, \ \omega_3 = 18$$

Amazingly, Ramanujan calculated all 18 examples when $g = 3$, a fact indicating that this manuscript was *not* written in the last year of his life when he was running out of time and left projects only partially completed in his lost notebook. For all these examples, by (15.3.7), we know that

$$X_1(s)X_2(1-s) = \frac{\pi}{32}.$$

Of the 20 possible choices for a_1, a_2, and a_3, the choices 1, 3, 5 and 0, 2, 4 yield the same inverse Mellin transforms that were obtained when $g = 1$, and so 18 remain to be examined. In all the examples recorded by Ramanujan, he replaced the variable x by $2x$ and multiplied the inverse Mellin transform by a constant (either ± 4 or ± 8). By the discourse after (15.2.6), and in particular, by (15.2.7), these changes alter the value of λ, and consequently, instead of obtaining the value $\sqrt{\pi/32}$, we obtain either the value $\sqrt{\pi}$ or $\frac{1}{2}\sqrt{\pi}$.

We work out the first of Ramanujan's 18 examples in detail. Since the calculations are similar for the remaining examples, we provide only the evaluations of the associated inverse Mellin transforms.

1. Set $a_1 = 0$, $a_2 = 1$, and $a_3 = 2$. Then

$$\begin{aligned}
\chi_1(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\sin\frac{\pi s}{6}\sin\frac{\pi(s+1)}{6}\sin\frac{\pi(s+2)}{6}x^{-s}ds \\
&= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\left\{\cos\frac{\pi}{6} - \cos\left(\frac{\pi s}{3}+\frac{\pi}{6}\right)\right\}\sin\frac{\pi(s+2)}{6}x^{-s}ds \\
&= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\left\{\frac{\sqrt{3}}{2}\left[\sin\frac{\pi s}{6}\cos\frac{\pi}{3}+\cos\frac{\pi s}{6}\sin\frac{\pi}{3}\right]\right. \\
&\qquad\qquad \left. -\frac{1}{2}\left[\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)+\sin\left(-\frac{\pi s}{6}+\frac{\pi}{6}\right)\right]\right\}x^{-s}ds \\
&= \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\left\{\sqrt{3}\sin\frac{\pi s}{6}+\cos\frac{\pi s}{6}-\cos\frac{\pi s}{2}\right\}x^{-s}ds \\
&= \frac{\sqrt{3}}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2}+\frac{1}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}-\frac{1}{4}\cos x,
\end{aligned}$$

where we have employed the evaluations (15.3.10), (15.3.11), and (15.3.9).

Next, it is easily checked that $\chi_2(x) = \chi_1(x)$, and so by the calculation above,

$$\chi_2(x) = \frac{\sqrt{3}}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2}+\frac{1}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}-\frac{1}{4}\cos x.$$

This is the first example recorded by Ramanujan under the heading $g = 3$, $\omega_3 = 18$ on page 227 of [269]. However, Ramanujan wrote

$$\lambda = \frac{1}{2}\sqrt{\pi}, \qquad \chi_1(x) = \chi_2(x) = \cos(2x) - e^{-x\sqrt{3}}\left(\cos x + \sqrt{3}\sin x\right). \quad (15.5.4)$$

Thus, Ramanujan replaced x by $2x$ and multiplied both $\chi_1(x)$ and $\chi_2(x)$ by -4. According to the discussion after (15.2.6), then $\lambda = \sqrt{\pi/32}$ is to be multiplied by $\sqrt{-4\cdot -4/2} = \sqrt{8}$, i.e., that $\lambda = \sqrt{\pi/32}$ is to be replaced by $\frac{1}{2}\sqrt{\pi}$.

As indicated above, we provide only a skeleton of the calculations for the remaining 17 examples. We first consider the five remaining cases in which

$\chi_1(x) = \chi_2(x)$. Second, we consider the 12 cases in which $\chi_1(x) \neq \chi_2(x)$. However, because the roles of $\chi_1(x)$ and $\chi_2(x)$ can be inverted, we only need to consider six cases. Also, because the same calculation with trigonometric functions is used repeatedly for different values of a_1, a_2, and a_3, we derive here in a very elementary fashion a general trigonometric identity that is to be used in all the examples that follow.

Let a, b, and c be any complex numbers. Then, by elementary trigonometry,

$$\begin{aligned}
F(a,b,c) :&= \sin\frac{\pi(s+a)}{6}\sin\frac{\pi(s+b)}{6}\sin\frac{\pi(s+c)}{6} \qquad (15.5.5)\\
&= \frac{1}{2}\left\{\cos\frac{\pi(a-b)}{6} - \cos\left(\frac{\pi(a+b)}{6}+\frac{\pi s}{3}\right)\right\}\sin\frac{\pi(s+c)}{6}\\
&= \frac{1}{2}\cos\frac{\pi(a-b)}{6}\left\{\cos\frac{\pi c}{6}\sin\frac{\pi s}{6}+\sin\frac{\pi c}{6}\cos\frac{\pi s}{6}\right\}\\
&\quad -\frac{1}{4}\left\{\sin\left(\frac{\pi(a+b+c)}{6}+\frac{\pi s}{2}\right)+\sin\left(\frac{\pi(c-a-b)}{6}-\frac{\pi s}{6}\right)\right\}\\
&= \frac{1}{2}\cos\frac{\pi(a-b)}{6}\left\{\cos\frac{\pi c}{6}\sin\frac{\pi s}{6}+\sin\frac{\pi c}{6}\cos\frac{\pi s}{6}\right\}\\
&\quad -\frac{1}{4}\left\{\cos\frac{\pi(a+b+c)}{6}\sin\frac{\pi s}{2}+\sin\frac{\pi(a+b+c)}{6}\cos\frac{\pi s}{2}\right.\\
&\qquad \left. -\cos\frac{\pi(a+b-c)}{6}\sin\frac{\pi s}{6}-\sin\frac{\pi(a+b-c)}{6}\cos\frac{\pi s}{6}\right\}.
\end{aligned}$$

We shall repeatedly employ (15.5.5) below without comment. Also, define, for $j=1,2$ and a, b, and c real,

$$\begin{aligned}
&\chi_j(a,b,c;x) := \chi_j(x)\\
&:= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\sin\frac{\pi(s+a)}{6}\sin\frac{\pi(s+b)}{6}\sin\frac{\pi(s+c)}{6}x^{-s}ds. \qquad (15.5.6)
\end{aligned}$$

Below, we repeatedly use (15.5.6) to calculate $\chi_1(x)$ and $\chi_2(x)$.

2. Let $a_1 = 3$, $a_2 = 4$, and $a_3 = 5$. Then

$$F(3,4,5) = -\frac{1}{4}\sin\frac{\pi s}{2} - \frac{1}{4}\sin\frac{\pi s}{6} + \frac{\sqrt{3}}{4}\cos\frac{\pi s}{6}$$

and, by (15.3.8), (15.3.10), and (15.3.11),

$$\chi_1(x) = \chi_2(x) = -\frac{1}{4}\sin x - \frac{1}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{\sqrt{3}}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

With the changes of variable mentioned above, $\lambda = \frac{1}{2}\sqrt{\pi}$.

3. Let $a_1 = 2$, $a_2 = 4$, and $a_3 = 5$. Then

$$F(2,4,5) = \frac{1}{8}\cos\frac{\pi s}{2} - \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} + \frac{1}{4}\cos\frac{\pi s}{6}$$

and, by (15.3.9), (15.3.8), and (15.3.11),

$$\chi_1(x) = \chi_2(x) = \frac{1}{8}\cos x - \frac{\sqrt{3}}{8}\sin x + \frac{1}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

In this case, the changes of variable yield $\lambda = \sqrt{\pi}$.

4. Let $a_1 = 1$, $a_2 = 2$, and $a_3 = 5$. Then

$$F(1,2,5) = \frac{1}{8}\sin\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} - \frac{1}{4}\sin\frac{\pi s}{6}$$

and, by (15.3.8), (15.3.9), and (15.3.10),

$$\chi_1(x) = \chi_2(x) = \frac{1}{8}\sin x + \frac{\sqrt{3}}{8}\cos x - \frac{1}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2}.$$

As above, the changes of variable yield $\lambda = \sqrt{\pi}$.

5. Let $a_1 = 0$, $a_2 = 3$, and $a_3 = 4$. Then

$$F(0,3,4) = \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} + \frac{1}{8}\cos\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\sin\frac{\pi s}{6} - \frac{1}{8}\cos\frac{\pi s}{6}.$$

Thus, by (15.3.8)–(15.3.11),

$$\chi_1(x) = \chi_2(x) = \frac{\sqrt{3}}{8}\sin x + \frac{1}{8}\cos x + \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} - \frac{1}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

As above, the changes of variable give $\lambda = \sqrt{\pi}$.

6. Let $a_1 = 0$, $a_2 = 1$, and $a_3 = 3$. Then

$$F(0,1,3) = \frac{1}{8}\sin\frac{\pi s}{2} - \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} + \frac{1}{8}\sin\frac{\pi s}{6} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{6}.$$

Hence, from (15.3.8), (15.3.9), (15.3.10), and (15.3.11),

$$\chi_1(x) = \chi_2(x) = \frac{1}{8}\sin x - \frac{\sqrt{3}}{8}\cos x + \frac{1}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

As before, the changes of variable give $\lambda = \sqrt{\pi}$.

The next six examples are those recorded by Ramanujan when $\chi_1(x) \neq \chi_2(x)$. We preserve the order in which Ramanujan gave the examples.

7. Let $a_1 = 0$, $a_2 = 4$, and $a_3 = 5$. Then

$$F(0,4,5) = \frac{1}{4}\cos\frac{\pi s}{2} + \frac{\sqrt{3}}{4}\sin\frac{\pi s}{6} - \frac{1}{4}\cos\frac{\pi s}{6}.$$

Hence, from (15.3.9), (15.3.10), and (15.3.11),

$$\chi_1(x) = \frac{1}{4}\cos x + \frac{\sqrt{3}}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2} - \frac{1}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Let $a_1 = 2$, $a_2 = 3$, and $a_3 = 4$. Then

$$F(2,3,4) = \frac{1}{4}\cos\frac{\pi s}{2} + \frac{1}{2}\cos\frac{\pi s}{6}.$$

Hence, from (15.3.9) and (15.3.11),

$$\chi_2(x) = \frac{1}{4}\cos x + \frac{1}{2}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

If we make the needed changes of variable, we find that $\lambda = \frac{1}{2}\sqrt{\pi}$.

8. Let $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. Then

$$F(1,2,3) = \frac{1}{4}\sin\frac{\pi s}{2} + \frac{1}{4}\sin\frac{\pi s}{6} + \frac{\sqrt{3}}{4}\cos\frac{\pi s}{6}.$$

Hence, by (15.3.8), (15.3.10), and (15.3.11),

$$\chi_1(x) = \frac{1}{4}\sin x + \frac{1}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{\sqrt{3}}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Let $a_1 = 0$, $a_2 = 1$, and $a_3 = 5$. Then

$$F(0,1,5) = \frac{1}{4}\sin\frac{\pi s}{2} - \frac{1}{2}\sin\frac{\pi s}{6}.$$

Hence, from (15.3.8) and (15.3.10),

$$\chi_2(x) = \frac{1}{4}\sin x - \frac{1}{2}e^{-\sqrt{3}x/2}\sin\frac{x}{2}.$$

The requisite changes of variable show us that $\lambda = \frac{1}{2}\sqrt{\pi}$.

9. Let $a_1 = 1$, $a_2 = 2$, and $a_3 = 4$. Then

$$F(1,2,4) = \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} + \frac{1}{8}\cos\frac{\pi s}{2} + \frac{1}{4}\cos\frac{\pi s}{6}.$$

Therefore, by (15.3.8)–(15.3.11),

$$\chi_1(x) = \frac{\sqrt{3}}{8}\sin x + \frac{1}{8}\cos x + \frac{1}{4}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Let $a_1 = 0$, $a_2 = 2$, and $a_3 = 5$. Then

$$F(0,2,5) = \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} + \frac{1}{8}\cos\frac{\pi s}{2} - \frac{\sqrt{3}}{8}\sin\frac{\pi s}{6} - \frac{1}{8}\cos\frac{\pi s}{6}.$$

Hence, from (15.3.8), (15.3.9), (15.3.10), and (15.3.11),

$$\chi_2(x) = \frac{\sqrt{3}}{8}\sin x + \frac{1}{8}\cos x - \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} - \frac{1}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

If we make the proper changes of variable, we find that $\lambda = \sqrt{\pi}$.

10. Let $a_1 = 1$, $a_2 = 4$, and $a_3 = 5$. Then

$$F(1,4,5) = -\frac{1}{8}\sin\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} + \frac{1}{4}\sin\frac{\pi s}{6}.$$

Hence, by (15.3.8)–(15.3.10),

$$\chi_1(x) = -\frac{1}{8}\sin x + \frac{\sqrt{3}}{8}\cos x + \frac{1}{4}e^{-\sqrt{3}x/2}\sin\frac{x}{2}.$$

Let $a_1 = 2$, $a_2 = 3$, and $a_3 = 5$. Then

$$F(2,3,5) = -\frac{1}{8}\sin\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} - \frac{1}{8}\sin\frac{\pi s}{6} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{6}.$$

So, from (15.3.8), (15.3.9), (15.3.10), and (15.3.11),

$$\chi_2(x) = -\frac{1}{8}\sin x + \frac{\sqrt{3}}{8}\cos x - \frac{1}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

The requisite changes of variable yield $\lambda = \sqrt{\pi}$.

11. Let $a_1 = 0$, $a_2 = 1$, and $a_3 = 4$. Then

$$F(0,1,4) = \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} - \frac{1}{8}\cos\frac{\pi s}{2} - \frac{\sqrt{3}}{8}\sin\frac{\pi s}{6} + \frac{1}{8}\cos\frac{\pi s}{6}.$$

Hence, by (15.3.8)–(15.3.11),

$$\chi_1(x) = \frac{\sqrt{3}}{8}\sin x - \frac{1}{8}\cos x - \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{1}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Let $a_1 = 0$, $a_2 = 2$, and $a_3 = 3$. Then

$$F(0,2,3) = \frac{\sqrt{3}}{8}\sin\frac{\pi s}{2} - \frac{1}{8}\cos\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\sin\frac{\pi s}{6} + \frac{1}{8}\cos\frac{\pi s}{6}.$$

Thus, from (15.3.8), (15.3.9), (15.3.10), and (15.3.11),

$$\chi_2(x) = \frac{\sqrt{3}}{8}\sin x - \frac{1}{8}\cos x + \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{1}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Making the same changes of variable as before, we see that $\lambda = \sqrt{\pi}$.

12. Let $a_1 = 0$, $a_2 = 3$, and $a_3 = 5$. Then

$$F(0,3,5) = \frac{1}{8}\sin\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} + \frac{1}{8}\sin\frac{\pi s}{6} - \frac{\sqrt{3}}{8}\cos\frac{\pi s}{6}.$$

Hence, by (15.3.8)–(15.3.11),

$$\chi_1(x) = \frac{1}{8}\sin x + \frac{\sqrt{3}}{8}\cos x + \frac{1}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} - \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

Let $a_1 = 1$, $a_2 = 3$, and $a_3 = 4$. Then

$$F(1,3,4) = \frac{1}{8}\sin\frac{\pi s}{2} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{2} + \frac{1}{8}\sin\frac{\pi s}{6} + \frac{\sqrt{3}}{8}\cos\frac{\pi s}{6}.$$

Thus, from (15.3.8), (15.3.9), (15.3.10), and (15.3.11),

$$\chi_2(x) = \frac{1}{8}\sin x + \frac{\sqrt{3}}{8}\cos x + \frac{1}{8}e^{-\sqrt{3}x/2}\sin\frac{x}{2} + \frac{\sqrt{3}}{8}e^{-\sqrt{3}x/2}\cos\frac{x}{2}.$$

As in the previous three examples, $\lambda = \sqrt{\pi}$.

16

A Preliminary Version of Ramanujan's Paper "On the Product $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$"

16.1 Introduction

The first four sections of this partial manuscript, which is found on pages 313–317 in Ramanujan's lost notebook [269], are almost identical to the first four sections of [254], [267, pp. 50–52]. We therefore feel that it is not necessary to offer further comments on these sections. Hence, the manuscript is copied here as it is printed in [269, pp. 313–315], except that we economize notation by using product and summation signs, instead of writing out the first few terms of a product or sum, as Ramanujan usually did. To aid readers, we occasionally insert remarks in square brackets. Section 5, however, is not included in [254]. Three of the equalities in this aborted section are incorrect, but they are easily corrected. One of the identities in this section is an expansion normally established with the use of partial fractions. Although the identity is correct, Ramanujan might have had doubts about his proofs of partial fraction expansions, because in an unpublished manuscript, which we thoroughly examined in Chap. 12, Ramanujan stated an incorrect partial fraction expansion prefaced by the assertion that he established it by the calculus of residues. Therefore, possibly because of a lack of confidence, Ramanujan chose not to include this section in his paper. Because this portion of the manuscript was not discussed in [254], we offer proofs for all the (corrected) claims. We remark that [254] might be considered to be a forerunner for Ramanujan's paper [255]. See also Entry 22 in Chap. 13 of Ramanujan's second notebook [268], [38, p. 225] and a paper by M. Chamberland and A. Straub [87] in which more general infinite products are examined.

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16.2 An Elegant Product Formula

Let

$$\phi(\alpha,\beta)=\prod_{n=1}^{\infty}\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}.$$

It can easily be shown that

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}=\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{2\alpha+\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}$$
$$\times\left\{1-\left(\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right)^2\right\}\left\{1-\left(\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right)^2\right\};\tag{16.2.1}$$

$$\prod_{n=1}^{\infty}\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{2\alpha+\beta}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)};\tag{16.2.2}$$

$$\prod_{n=1}^{\infty}\left\{1-\left(\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right)^2\right\}\left\{1-\left(\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right)^2\right\}$$
$$=\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}.\tag{16.2.3}$$

It follows from (16.2.1)–(16.2.3) that

$$\phi(\alpha,\beta)\phi(\beta,\alpha)$$
$$=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\cdot\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}.\tag{16.2.4}$$

But $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms if $\alpha-\beta$ be any integer. It follows from (16.2.4) that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)$ can be expressed in finite terms. That is to say

$$\prod_{k=1}^{\infty}\left\{1+\left(\frac{x}{n+k}\right)^3\right\}\tag{16.2.5}$$

can be expressed in finite terms, if $x-2n$ be any integer.

(Because of careless photocopying by the publisher, a portion of the right side of (16.2.4) and part of the discourse between (16.2.4) and (16.2.5) were cropped. The product in [254] corresponding to (16.2.5) appears to be more general, but it is easy to see that with simple changes of variables, the two formulations have the same generality.)

16.3 The Special Case $\alpha = \beta$

Suppose now that $\alpha = \beta$ in (16.2.4). We obtain

$$\prod_{n=1}^{\infty} \left\{ 1 + \left(\frac{2\alpha}{n+\alpha} \right)^3 \right\} = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)} \cdot \frac{\sinh \pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \tag{16.3.1}$$

Similarly supposing that $\beta = \alpha + 1$ in (16.2.4) we obtain

$$\prod_{n=1}^{\infty} \left\{ 1 + \left(\frac{1+2\alpha}{n+\alpha} \right)^3 \right\} = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)} \cdot \frac{\cosh \pi(\frac{1}{2}+\alpha)\sqrt{3}}{\pi}. \tag{16.3.2}$$

Since

$$\left\{ 1 + \left(\frac{\alpha}{n} \right)^3 \right\} \left\{ 1 + 3\left(\frac{\alpha}{2n+\alpha} \right)^2 \right\} = \frac{1 + \dfrac{\alpha}{n}}{\left(1 + \dfrac{\alpha}{2n} \right)^2} \left(1 + \frac{\alpha^2}{n^2} + \frac{\alpha^4}{n^4} \right)$$

it is easy to see that

$$\prod_{n=1}^{\infty} \left(1 + \left(\frac{\alpha}{n} \right)^3 \right) \left\{ 1 + 3\left(\frac{\alpha}{2n+\alpha} \right)^2 \right\} = \frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}(1+\alpha))} \cdot \frac{\cosh \pi\alpha\sqrt{3} - \cos \pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}. \tag{16.3.3}$$

16.4 An Application of Binet's Formula

It is known that, if the real part of α is positive, then

$$\log \Gamma(\alpha) = \left(\alpha - \tfrac{1}{2} \right) \log \alpha - \alpha + \tfrac{1}{2} \log(2\pi) + 2 \int_0^{\infty} \frac{\tan^{-1}(x/\alpha)}{e^{2\pi x} - 1} dx. \tag{16.4.1}$$

(The representation for $\log \Gamma(\alpha)$ in (16.4.1) is known as Binet's integral formula for $\log \Gamma(\alpha)$, and a proof can be found in [315, p. 251].) From this we can easily show that, if the real part of α is positive, then

$$\begin{aligned} &\frac{1}{2} \log(2\pi\alpha) + \log \left\{ \prod_{n=1}^{\infty} \left(1 + \left(\frac{\alpha}{n} \right)^3 \right) \right\} \\ &\quad = \log \left(\frac{\cosh \pi\alpha\sqrt{3} - \cos \pi\alpha}{\pi\alpha} \right) - \frac{\pi\alpha}{\sqrt{3}} + 2 \int_0^{\infty} \frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x} - 1} dx. \end{aligned} \tag{16.4.2}$$

Hence we see that

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi\alpha x}-1}dx$$

can be expressed in finite terms for all positive integral values of α. Thus for example

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi x}-1}dx = \frac{1}{4}\log 2\pi - \frac{1}{2}\log\left(1+e^{-\pi\sqrt{3}}\right) - \frac{\pi}{4\sqrt{3}};$$
$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{4\pi x}-1}dx = \frac{1}{8}\log 12\pi - \frac{1}{4}\log\left(1+e^{-2\pi\sqrt{3}}\right) - \frac{\pi}{4\sqrt{3}};$$

and so on.

16.5 A Sum–Integral Identity

It is easy to see that

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}n^2}{n^3+\alpha^3} = \frac{1}{3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n+\alpha} + \frac{4}{3}\sum_{n=1}^\infty \frac{(-1)^{n-1}(2n-\alpha)}{(2n-\alpha)^2+3\alpha^2}. \tag{16.5.1}$$

Since

$$\sum_{n=0}^\infty \frac{(-1)^n(2n+1)}{(2n+1)^2+x^2} = \frac{\pi}{4}\operatorname{sech}\frac{\pi x}{2},$$

we see that the left-hand side of (16.5.1) can be expressed in finite terms if α is any odd integer. For example

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}n^2}{n^3+1} = \frac{1}{3}\left(1-\log 2+\pi\operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right).$$

Again, if $\alpha > 0$, then

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{\alpha^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2} + \sum_{n=1}^\infty \frac{(-1)^n}{n^2+x^2}\right\}\frac{x^6 dx}{\alpha^6+x^6}$$
$$= \frac{1}{3}\sum_{n=0}^\infty \frac{(-1)^n}{n+\alpha} - \frac{4}{3}\sum_{n=1}^\infty \frac{(-1)^{n-1}(2n+\alpha)}{(2n+\alpha)^2+3\alpha^2}. \tag{16.5.2}$$

Hence the left-hand side of (16.5.2) can be expressed in finite terms if α is any odd integer. For example

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\cdot\frac{dx}{1+x^6} = \frac{1}{3}\left(\log 2 - 1+\pi\operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right).$$

16.6 The Unpublished Section

(We emphasize that certain identities in this section are incorrect. Proofs of all correct and corrected identities in Sect. 16.6 are provided in Sect. 16.7.) If 2α be a positive integer, then it can easily be shown that

$$\sum_{n=1}^{\infty} \frac{n}{n^4+4\alpha^4} = \frac{1}{4\alpha} \sum_{n=1}^{2\alpha} \frac{1}{(n-\alpha)^2+\alpha^2}; \tag{16.6.1}$$

and

$$\int_0^{\infty} \frac{x}{e^{2\pi x}-1} \cdot \frac{dx}{4\alpha^4+x^4} = \frac{\pi}{8\alpha^2} \frac{(-1)^{2\alpha}}{e^{2\pi\alpha}+(-1)^{2\alpha+1}} + \frac{1}{8\alpha} \sum_{n=1}^{2\alpha} \frac{1}{(n-\alpha)^2+\alpha^2}. \tag{16.6.2}$$

It can easily be shown by the theory of residues, that

$$\frac{1}{16\pi\alpha^4} + \sum_{n=1}^{\infty} \frac{n \coth n\pi}{n^4+4\alpha^4} = \frac{\pi}{8\alpha^2} \cdot \frac{\cosh 2\pi\alpha + \cos 2\pi\alpha}{\cosh 2\pi\alpha - \cos 2\pi\alpha}. \tag{16.6.3}$$

It follows from (16.6.1) and (16.6.3) that, if 2α be a positive integer, then

$$\sum_{n=1}^{\infty} \frac{n}{e^{2n\pi}-1} \frac{1}{n^4+4\alpha^4} = \frac{\pi}{16\alpha^2} \left\{ \frac{e^{2\pi\alpha}+(-1)^{2\alpha}}{e^{2\pi\alpha}-(-1)^{2\alpha}} \right\}^2 - \frac{1}{32\pi\alpha^4} - \frac{1}{8\alpha} \sum_{n=1}^{2\alpha} \frac{1}{(n-\alpha)^2+\alpha^2}. \tag{16.6.4}$$

In a similar manner we can show that, if α be a positive integer, then

$$\sum_{n=0}^{\infty} \frac{2n+1}{(2n+1)^4+4\alpha^4} = \frac{1}{4\alpha} \sum_{n=0}^{\alpha-1} \frac{1}{(2n+1-\alpha)^2+\alpha^2}; \tag{16.6.5}$$

and

$$\sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)\pi}+1} \cdot \frac{1}{(2n+1)^4+4\alpha^4} = \frac{\pi}{32\alpha^2} \left\{ \frac{e^{\pi\alpha}-(-1)^{\alpha}}{e^{\pi\alpha}+(-1)^{\alpha}} \right\}^2 - \frac{1}{8\alpha} \sum_{n=0}^{\alpha-1} \frac{1}{(2n+1-\alpha)^2+\alpha^2}. \tag{16.6.6}$$

It follows from (16.6.4) that, if 2α be a positive integer, then

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{e^{2n\pi}-1} - \frac{1}{e^{2\pi(n+2\alpha)}-1} \right\} \frac{1}{(n+\alpha)^2+\alpha^2}$$

can be expressed in finite terms. Similarly from (16.6.6) we see that, if α be a positive integer, then

$$\sum_{n=1}^{\infty}\left\{\frac{1}{e^{(2n-1)\pi}+1}-\frac{1}{e^{(2\alpha+2n-1)\pi}+1}\right\}\frac{1}{(2n-1+\alpha)^2+\alpha^2}$$

can be expressed in finite terms. For example

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+(n+1)^2)(\sinh(2n+1)\pi-\sinh\pi)} = \frac{1}{2\sinh\pi}\left(\frac{1}{\pi}+\coth\pi-\frac{\pi}{2}\tanh^2\frac{\pi}{2}\right). \qquad (16.6.7)$$

16.7 Proofs of the Equalities in Sect. 16.6

Proof of (16.6.1). Observe that

$$\sum_{n=1}^{\infty}\frac{n}{n^4+4\alpha^4}=\frac{1}{4\alpha}\sum_{n=1}^{\infty}\frac{1}{(n-\alpha)^2+\alpha^2}-\frac{1}{4\alpha}\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^2+\alpha^2}. \qquad (16.7.1)$$

If we now assume that 2α is a positive integer, we see that all the terms in the second series on the right-hand side of (16.7.1) are canceled by those in the first series. Since the largest index for those terms in the first series that are not canceled by those in the second series is $n = 2\alpha$, the identity (16.6.1) follows. □

Proof of a Corrected Version of (16.6.2). The identity (16.6.2) should be replaced by

$$\int_0^{\infty}\frac{x}{e^{2\pi x}-1}\cdot\frac{dx}{4\alpha^4+x^4}=\frac{\pi}{8\alpha^2}\frac{\cos(2\pi\alpha)-e^{2\pi\alpha}}{\cosh(2\pi\alpha)-\cos(2\pi\alpha)} + \frac{\pi}{16\alpha^2}-\frac{1}{8\alpha}\sum_{n=1}^{2\alpha}\frac{1}{(n-\alpha)^2+\alpha^2}. \qquad (16.7.2)$$

In fact, on page 269 in his second notebook [268], [41, p. 419, Entry 8], Ramanujan states a more general formula,

$$\sum_{n=1}^{\infty}\frac{n^{m+1}}{n^4+4\alpha^4}=\frac{\pi}{4}(\alpha\sqrt{2})^{m-2}\sec\left(\tfrac{1}{4}\pi m\right) - 2\cos\left(\tfrac{1}{2}\pi m\right)\int_0^{\infty}\frac{x^{m+1}dx}{(e^{2\pi x}-1)(x^4+4\alpha^4)} + \frac{\pi}{2}(\alpha\sqrt{2})^{m-2}\frac{\cos\left(\frac{1}{4}\pi m+2\pi\alpha\right)-e^{2\pi\alpha}\cos\left(\frac{1}{4}\pi m\right)}{\cosh(2\pi\alpha)-\cos(2\pi\alpha)}, \qquad (16.7.3)$$

where m is a nonnegative integer. Note that if we set $m = 0$ in (16.7.3), we obtain (16.7.2) upon the use of (16.6.1). □

The identity (16.6.2), or the corrected version (16.7.2), is actually not further used in this fragment. Possibly, Ramanujan recorded it because he considered (16.6.2) to be an integral analogue of (16.6.4).

Proof of (16.6.3). This partial fraction decomposition can be found as a corollary in Sect. 20 of Chap. 14 in Ramanujan's second notebook [268], [38, p. 274]. □

Proof of (16.6.4). First, since 2α is a positive integer, elementary manipulation easily shows that

$$\frac{\cosh(2\pi\alpha) + \cos(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\alpha)} = \left\{ \frac{e^{2\pi\alpha} + (-1)^{2\alpha}}{e^{2\pi\alpha} - (-1)^{2\alpha}} \right\}^2. \tag{16.7.4}$$

Thus, by (16.7.4), (16.6.3), and (16.6.1),

$$\begin{aligned}
\frac{\pi}{8\alpha^2} \left\{ \frac{e^{2\pi\alpha} + (-1)^{2\alpha}}{e^{2\pi\alpha} - (-1)^{2\alpha}} \right\}^2 &= \frac{1}{16\pi\alpha^4} + \sum_{n=1}^{\infty} \frac{n \coth n\pi}{n^4 + 4\alpha^4} \\
&= \frac{1}{16\pi\alpha^4} + \sum_{n=1}^{\infty} \left\{ 1 + \frac{2}{e^{2n\pi} - 1} \right\} \frac{n}{n^4 + 4\alpha^4} \\
&= \frac{1}{16\pi\alpha^4} + 2\sum_{n=1}^{\infty} \frac{n}{e^{2n\pi} - 1} \frac{1}{n^4 + 4\alpha^4} + \frac{1}{4\alpha} \sum_{n=1}^{2\alpha} \frac{1}{(n-\alpha)^2 + \alpha^2}.
\end{aligned}$$

Rearranging the last identity yields (16.6.4). □

Proof of (16.6.5). We can easily see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2n+1}{(2n+1)^4 + 4\alpha^4} = \frac{1}{4\alpha} &\sum_{n=0}^{\infty} \frac{1}{(2n+1-\alpha)^2 + \alpha^2} \\
&- \frac{1}{4\alpha} \sum_{n=0}^{\infty} \frac{1}{(2n+1+\alpha)^2 + \alpha^2}.
\end{aligned} \tag{16.7.5}$$

The terms in the two series on the right side of (16.7.5) cancel each other, except for the first α terms of the first series, and so the identity (16.6.5) follows. □

Proof of a Corrected Version of (16.6.6). From Entry 25 in Chap. 14 of Ramanujan's second notebook [268], [38, p. 292],

$$\sum_{n=0}^{\infty} \frac{2n+1}{(e^{(2n+1)\pi}+1)((2n+1)^4+4\alpha^4)} = \frac{\pi}{32\alpha^2} - \frac{\pi e^{-\pi\alpha}}{16\alpha^2(\cosh(\pi\alpha)+\cos(\pi\alpha))} - \frac{1}{4\alpha}\sum_{n=0}^{\infty}\frac{1}{(2n+1+\alpha)^2+\alpha^2}. \tag{16.7.6}$$

Replacing α by $-\alpha$ in (16.7.6), we find that

$$\sum_{n=0}^{\infty} \frac{2n+1}{(e^{(2n+1)\pi}+1)((2n+1)^4+4\alpha^4)} = \frac{\pi}{32\alpha^2} - \frac{\pi e^{\pi\alpha}}{16\alpha^2(\cosh(\pi\alpha)+\cos(\pi\alpha))} + \frac{1}{4\alpha}\sum_{n=0}^{\infty}\frac{1}{(2n+1-\alpha)^2+\alpha^2}. \tag{16.7.7}$$

Adding (16.7.6) and (16.7.7), dividing both sides by 2, and observing the same cancellation as previously noted in (16.7.5), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{2n+1}{(e^{(2n+1)\pi}+1)((2n+1)^4+4\alpha^4)} \\ &= \frac{\pi}{32\alpha^2} - \frac{\pi}{16\alpha^2}\cdot\frac{\cosh(\pi\alpha)}{\cosh(\pi\alpha)+\cos(\pi\alpha)} + \frac{1}{8\alpha}\sum_{n=0}^{\alpha-1}\frac{1}{(2n+1-\alpha)^2+\alpha^2} \\ &= -\frac{\pi}{32\alpha^2}\cdot\left\{\frac{e^{\pi\alpha}-(-1)^\alpha}{e^{\pi\alpha}+(-1)^\alpha}\right\}^2 + \frac{1}{8\alpha}\sum_{n=0}^{\alpha-1}\frac{1}{(2n+1-\alpha)^2+\alpha^2}. \end{aligned}\tag{16.7.8}$$

Comparing (16.7.8) with (16.6.6), we see that Ramanujan's claim (16.6.6) can be corrected by multiplying the right-hand side by -1. □

We now justify the two claims made by Ramanujan below (16.6.6). Recalling that 2α is a positive integer, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} &\left\{\frac{1}{e^{2n\pi}-1} - \frac{1}{e^{2(n+2\alpha)\pi}-1}\right\}\frac{1}{(n+\alpha)^2+\alpha^2} \\ &= \sum_{n=1}^{\infty}\left\{\frac{1}{(e^{2n\pi}-1)((n+\alpha)^2+\alpha^2)} - \frac{1}{(e^{2n\pi}-1)((n-\alpha)^2+\alpha^2)}\right\} \\ &\quad + \sum_{n=1}^{2\alpha}\frac{1}{(e^{2n\pi}-1)((n-\alpha)^2+\alpha^2)}\end{aligned}$$

$$= -\sum_{n=1}^{\infty} \frac{4n\alpha}{(e^{2n\pi}-1)(n^4+4\alpha^4)} + \sum_{n=1}^{2\alpha} \frac{1}{(e^{2n\pi}-1)((n-\alpha)^2+\alpha^2)}$$

$$= -4\alpha \left\{ \frac{\pi}{16\alpha^2} \left\{ \frac{e^{2\pi\alpha}+(-1)^{2\alpha}}{e^{2\pi\alpha}-(-1)^{2\alpha}} \right\}^2 - \frac{1}{32\pi\alpha^4} - \frac{1}{8\alpha} \sum_{n=1}^{2\alpha} \frac{1}{(n-\alpha)^2+\alpha^2} \right\}$$

$$+ \sum_{n=1}^{2\alpha} \frac{1}{(e^{2n\pi}-1)((n-\alpha)^2+\alpha^2)}, \tag{16.7.9}$$

by (16.6.4). This justifies the first claim.

The proof of the second is similar. Now, recalling that α is a positive integer and using (16.7.8), we find that

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{e^{(2n-1)\pi}+1} - \frac{1}{e^{(2n+2\alpha-1)\pi}+1} \right\} \frac{1}{(2n-1+\alpha)^2+\alpha^2}$$

$$= \sum_{n=1}^{\infty} \left\{ \frac{1}{(e^{(2n-1)\pi}+1)((2n-1+\alpha)^2+\alpha^2)} \right.$$

$$\left. - \frac{1}{(e^{(2n-1)\pi}+1)((2n-1-\alpha)^2+\alpha^2)} \right\}$$

$$+ \sum_{n=1}^{\alpha} \frac{1}{(e^{(2n-1)\pi}+1)((2n-1-\alpha)^2+\alpha^2)}$$

$$= -\sum_{n=1}^{\infty} \frac{4(2n-1)\alpha}{(e^{(2n-1)\pi}+1)((2n-1)^4+4\alpha^4)}$$

$$+ \sum_{n=1}^{\alpha} \frac{1}{(e^{(2n-1)\pi}+1)((2n-1-\alpha)^2+\alpha^2)}$$

$$= -4\alpha \left\{ -\frac{\pi}{32\alpha^2} \cdot \left\{ \frac{e^{\pi\alpha}-(-1)^{\alpha}}{e^{\pi\alpha}+(-1)^{\alpha}} \right\}^2 + \frac{1}{8\alpha} \sum_{n=0}^{\alpha-1} \frac{1}{(2n+1-\alpha)^2+\alpha^2} \right\}$$

$$+ \sum_{n=1}^{\alpha} \frac{1}{(e^{(2n-1)\pi}+1)((2n-1-\alpha)^2+\alpha^2)}.$$

Proof of a Corrected Version of (16.6.7). Let $\alpha = \frac{1}{2}$ in (16.7.9) to deduce that

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{e^{2n\pi}-1} - \frac{1}{e^{(2n+2)\pi}-1} \right\} \frac{1}{n^2+n+\frac{1}{2}}$$

$$= \sum_{n=1}^{\infty} \frac{2\sinh\pi}{(\cosh(2n+1)\pi - \cosh\pi)(n^2+(n+1)^2)}$$

$$= -2\left\{\frac{\pi}{4}\left\{\frac{e^{\pi}-1}{e^{\pi}+1}\right\}^2 - \frac{1}{2\pi} - \frac{1}{2}\right\} + \frac{2}{e^{2\pi}-1}$$
$$= -\frac{\pi}{2}\tanh^2\frac{\pi}{2} + \frac{1}{\pi} + \coth\pi. \tag{16.7.10}$$

Dividing both sides of (16.7.10) by $2\sinh\pi$, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{(\cosh(2n+1)\pi - \cosh\pi)(n^2+(n+1)^2)}$$
$$= \frac{1}{2\sinh\pi}\left(-\frac{\pi}{2}\tanh^2\frac{\pi}{2} + \frac{1}{\pi} + \coth\pi\right).$$

Hence, in Ramanujan's claim (16.6.7), $\sinh(2n+1)\pi - \sinh\pi$ should be replaced by $\cosh(2n+1)\pi - \cosh\pi$. □

17

A Preliminary Version of Ramanujan's Paper "On the Integral $\int_0^x \frac{\tan^{-1} t}{t} dt$"

17.1 Introduction

The partial manuscript on pages 322–325 in Ramanujan's lost notebook [269] is a preliminary version of Ramanujan's seventh published paper [250], [267, pp. 40–43]. However, not all of the material in [250] can be found in this preliminary version. The photographer of the manuscript for [269] unfortunately inverted the order of the second and third pages. Moreover, the second page is written in rougher, less legible handwriting, indicating that this page is a replacement for a more legible page that has evidently been lost. The photographic reproduction of this page was so poor that we had to rely on Ramanujan's paper [250] to decipher several formulas. As in the manuscript discussed in Chap. 16, we faithfully copy what Ramanujan has written, except that we economize series and product notation, and for clarity we introduce parentheses in arguments of certain functions. At the end of the manuscript, we offer a few additional comments.

17.2 Ramanujan's Preliminary Manuscript

Let

$$\Phi(x) = \int_0^x \frac{\tan^{-1} t}{t} dt.$$

Then, changing t into $1/t$, it is easy to see that if $x > 0$,

$$\Phi(x) - \Phi\left(\frac{1}{x}\right) = \frac{1}{2}\pi \log x. \tag{17.2.1}$$

It is also clear that if $-1 \leq x \leq 1$, then

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}. \tag{17.2.2}$$

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The following results can easily be proved by differentiating both sides with respect to x.

If $0 < x < \frac{1}{2}\pi$, then

$$\sum_{n=0}^{\infty} \frac{\sin(4n+2)x}{(2n+1)^2} = \Phi(\tan x) - x \log \tan x. \tag{17.2.3}$$

As particular cases of (17.2.3) we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)^2} = \Phi(\sqrt{2}-1) + \frac{\pi}{8}\log(1+\sqrt{2}) + \frac{\pi^2}{16}; \tag{17.2.4}$$

$$\Phi(1) = \frac{3}{2}\Phi(2-\sqrt{3}) + \frac{1}{8}\pi \log(2+\sqrt{3}). \tag{17.2.5}$$

If $0 < x < \frac{1}{2}\pi$, then

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} \frac{\cos^{2n+1} x + \sin^{2n+1} x}{(2n+1)^2} = \Phi(\tan x) + \frac{1}{2}\pi \log(2\cos x); \tag{17.2.6}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \cos^n x = \Phi(\tan x) + \frac{1}{2}\pi \log\cos x - x \log \sin x. \tag{17.2.7}$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and 1 is greater than or equal to either $|(1-\alpha)\sin x|$ or $|(1-1/\alpha)\cos x|$, then

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \left(1 - \frac{1}{\alpha}\right)^n \cos^n x + \sum_{n=1}^{\infty} \frac{\sin(n\{x+\frac{1}{2}\pi\})}{n^2} (1-\alpha)^n \sin^n x$$
$$= \Phi(\tan x) - \Phi(\alpha \tan x) + x \log \alpha. \tag{17.2.8}$$

If $-1 < x < 1$, then

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) \log\left(1 - \frac{x^2}{(2n+1)^2}\right)$$
$$= \frac{4}{\pi}\left\{\Phi(1) - \Phi\left(\tan\{\tfrac{1}{4}\pi(1-x)\}\right)\right\} + \log\left(\tan\{\tfrac{1}{4}\pi(1-x)\}\right). \tag{17.2.9}$$

As an example we have

$$\prod_{n=0}^{\infty} \left(1 - \frac{4}{(6n+3)^2}\right)^{(-1)^n(2n+1)} = \frac{\exp\left(\dfrac{4}{3\pi}\Phi(1)\right)}{(2+\sqrt{3})^{2/3}}. \tag{17.2.10}$$

If x is real, then

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1+\frac{x^2}{(2n+1)^2}\right)$$
$$=\frac{4}{\pi}\left\{\Phi(1)-\Phi\left(e^{-\frac{1}{2}\pi x}\right)\right\}-2x\,\tan^{-1}\left(e^{-\frac{1}{2}\pi x}\right). \quad (17.2.11)$$

For example

$$\prod_{n=0}^{\infty}\left(1+\left\{\frac{2}{(2n+1)\pi}\log(2+\sqrt{3})\right\}^2\right)^{(-1)^n(2n+1)}=\exp\left(\frac{4}{3\pi}\Phi(1)\right). \quad (17.2.12)$$

It follows from (17.2.9) and (17.2.11) that, if $-1<\beta<1$ and $\frac{1}{2}\pi\alpha=\log\left(\tan\{\frac{1}{4}\pi(1+\beta)\}\right)$, then

$$\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^7\cdots=e^{\frac{1}{2}\pi\alpha\beta}. \quad (17.2.13)$$

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right)^{(-1)^n(2n+1)}=\frac{\pi}{8}\exp\left(\frac{4}{\pi}\Phi(1)\right). \quad (17.2.14)$$

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1+\frac{64x^4}{(2n+1)^4}\right)=\frac{8}{\pi}\Phi(1)-2x\log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)$$
$$-4x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)-\frac{8}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\cos(2n+1)\pi x}{(2n+1)^2}e^{-(2n+1)\pi x}. \quad (17.2.15)$$

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\tan^{-1}\frac{8x^2}{(2n+1)^2}=\log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)$$
$$-2x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)+\frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(2n+1)\pi x}{(2n+1)^2}e^{-(2n+1)\pi x}. \quad (17.2.16)$$

If n is a positive odd integer,

$$\prod_{k=0}^{\infty}\left(1+\frac{4n^4}{(2k+1)^4}\right)^{(-1)^k(2k+1)}=\left(\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right)^{2n(-1)^{\frac{1}{2}(n-1)}}\exp\left(\frac{8}{\pi}\Phi(1)\right). \quad (17.2.17)$$

If n is any even positive integer,

$$\prod_{k=0}^{\infty}\left(1+\frac{4n^4}{(2k+1)^4}\right)^{(-1)^k(2k+1)} \tag{17.2.18}$$
$$= \exp\left\{\frac{8}{\pi}\Phi(1) - \frac{8}{\pi}(-1)^{\frac{1}{2}n}\left(\Phi\left(e^{-\frac{1}{2}\pi n}\right) + \frac{1}{2}\pi n \tan^{-1}\left(e^{-\frac{1}{2}\pi n}\right)\right)\right\}.$$

If n is a positive odd integer,

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)\tan^{-1}\frac{2n^2}{(2k+1)^2} \tag{17.2.19}$$
$$= \frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right) + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}e^{-\frac{1}{2}(2k+1)\pi n}\right\}.$$

If n is a positive even integer,

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)\tan^{-1}\frac{2n^2}{(2k+1)^2} = n(-1)^{\frac{1}{2}(n-1)}\tan^{-1}\left(e^{-\frac{1}{2}\pi n}\right). \tag{17.2.20}$$

By the theory of residues it can be shown that, if α and β are positive and $\alpha\beta = \pi^2$, then

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(e^{(2n+1)\alpha}-1)} + \frac{\pi}{4\beta}\sum_{n=1}^{\infty}\frac{1}{n^2(e^{n\beta}+e^{-n\beta})} = \frac{\pi}{16}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right) - \frac{1}{2}\Phi(1). \tag{17.2.21}$$

Putting $\alpha = \beta = \pi$ in (17.2.21) we can easily calculate the value of $\Phi(1)$ approximately. Thus

$$\Phi(1) = 0.9159655942 \tag{17.2.22}$$

approximately.

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then

$$\sum_{n=0}^{\infty}\frac{n!}{(\frac{3}{2})_n}\frac{\sin^{2n+1}x}{2n+1} = 2\Phi(\tan\tfrac{1}{2}x). \tag{17.2.23}$$

17.3 Commentary

At the top of the first page of the manuscript, brief notes, possibly in Hardy's handwriting, are appended. In particular, it is mentioned that (17.2.23) is not given in Ramanujan's published paper [250]. However, (17.2.23) indeed can be found in Ramanujan's paper; see equation (8) there. As pointed out in Ramanujan's *Collected Papers* [267, pp. 40, 337], the identity (17.2.4) is incorrect, with the corrected version being

$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{(2n+1)^2} = \sqrt{2}\Phi(\sqrt{2}-1) + \frac{\pi}{4\sqrt{2}} \log(1+\sqrt{2}).$$

As remarked in the introduction, not all of the results in [250] can be found in this preliminary version. For the convenience of readers, we provide a table indicating the identity in [250] corresponding to the identity in this manuscript:

[269]	[250]	[269]	[250]	[269]	[250]
(17.2.1)	(4)	(17.2.9)	(12)	(17.2.17)	(20)
(17.2.2)	(3)	(17.2.10)	(13)	(17.2.18)	(21)
(17.2.3)	(5)	(17.2.11)	(15)	(17.2.19)	(22)
(17.2.4)	(6)	(17.2.12)	(16)	(17.2.20)	(23)
(17.2.5)	(7)	(17.2.13)	(17)	(17.2.21)	(25)
(17.2.6)	(10)	(17.2.14)	(14)	(17.2.22)	(27)
(17.2.7)	(9)	(17.2.15)	(18)	(17.2.23)	(8)
(17.2.8)	(11)	(17.2.16)	(19)		

18

A Partial Manuscript Connected with Ramanujan's Paper "Some Definite Integrals"

18.1 Introduction

A partial manuscript on definite integrals is found on pages 190–191 in [269]. The manuscript was intended to be Sect. 4 of a paper whose identity is unknown to us. The manuscript's content points to Ramanujan's paper "Some Definite Integrals" [255], [267, pp. 53–58]. The first sentence on page 190 and the conclusion of the manuscript might lead one to conclude that this fragment is connected with Ramanujan's paper "New Expressions for Riemann's Functions $\xi(s)$ and $\Xi(s)$", [257], [267, pp. 72–77], but the connections with the former paper appear stronger. The manuscript's ten integral formulas are numbered (18)–(27). In the next section, we copy the partial manuscript, and in the following section we offer commentary.

Page 192 in [269] provides a list of three Dirichlet L-series. Possibly Ramanujan briefly began here another section to be added to the partial manuscript on pages 190–191, because, as we shall see in the sequel, the integrals that are evaluated on these two pages have associations with L-series. Page 203 is an isolated page on which Ramanujan evaluates six quotients of either Riemann zeta functions or L-functions. Because of the bond with L-functions, we have also chosen to discuss this page in this chapter.

18.2 The Partial Manuscript

4. There are of course results corresponding to all the previous results for the functions analogous to the ζ-function. Thus, for example we have

$$\int_0^\infty \frac{\cos 2nx}{\cosh \pi x} dx = \frac{1}{2\cosh n}, \tag{18.2.18}$$

$$\int_0^\infty \frac{\cos 3nx}{1 + 2\cosh \pi x} dx = \frac{1}{\sqrt{3}} \cdot \frac{1}{1 + 2\cosh 2n}, \tag{18.2.19}$$

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$$\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} \cos 4nx \, dx = \frac{1}{2\sqrt{2}} \cdot \frac{\cosh n}{\cosh 2n}, \tag{18.2.20}$$

$$\int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x} \sin 4nx \, dx = \frac{1}{2\sqrt{2}} \cdot \frac{\sinh n}{\cosh 2n}, \tag{18.2.21}$$

$$\int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1} \sin 6nx \, dx = \frac{1}{2\sqrt{3}} \cdot \frac{\sinh n}{2\cosh 2n - 1}. \tag{18.2.22}$$

From these we can easily deduce that

$$\sqrt{\alpha} \int_0^\infty \frac{e^{-(\alpha x)^2}}{\cosh \pi x} dx = \sqrt{\beta} \int_0^\infty \frac{e^{-(\beta x)^2}}{\cosh \pi x} dx \tag{18.2.23}$$

with the condition that $\alpha\beta = \pi$.

$$\sqrt{\alpha} \int_0^\infty \frac{e^{-(\alpha x)^2}}{2\cosh \pi x + 1} dx = \sqrt{\beta} \int_0^\infty \frac{e^{-(\beta x)^2}}{2\cosh \pi x + 1} dx \tag{18.2.24}$$

with the condition that $\alpha\beta = \frac{3}{2}\pi$.

$$\sqrt{\alpha} \int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} e^{-(\alpha x)^2} dx = \sqrt{\beta} \int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} e^{-(\beta x)^2} dx \tag{18.2.25}$$

with the condition that $\alpha\beta = 2\pi$.

$$\sqrt{\alpha} \int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x} \alpha x e^{-(\alpha x)^2} dx = \sqrt{\beta} \int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x} \beta x e^{-(\beta x)^2} dx \tag{18.2.26}$$

with the condition that $\alpha\beta = 2\pi$.

$$\sqrt{\alpha} \int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1} \alpha x e^{-(\alpha x)^2} dx = \sqrt{\beta} \int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1} \beta x e^{-(\beta x)^2} dx \tag{18.2.27}$$

with the condition that $\alpha\beta = 3\pi$.

From the above results we can easily deduce the results corresponding to those of §§2–3 for the functions

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s},$$

$$\sum_{n=1}^\infty \left(\frac{n}{3}\right) \frac{1}{n^s},$$

$$\sum_{n=0}^\infty \frac{(-1)^{n(n-1)/2}}{(2n+1)^s},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{(2n+1)^s},$$

$$\sum_{n=0}^{\infty} \left(\frac{2n+1}{3}\right) \frac{(-1)^n}{(2n+1)^s},$$

where $1, 3, 5, 7, \dots$ are the natural odd numbers, $1, 2, 4, 5, \dots$ are the natural numbers without the multiples of 3 and $1, 5, 7, 11, \dots$ are the natural odd numbers without the multiples of 3.

18.3 Discussion and Proofs of the Identities

We have corrected two misprints. The condition $\alpha\beta = \frac{3}{2}\pi$ for (18.2.24) is a replacement of Ramanujan's for $\alpha\beta = \frac{3}{4}\pi$. On the right-hand side of (18.2.26), Ramanujan inadvertently wrote α at it first appearance in the integrand, instead of β.

The identities (18.2.18) and (18.2.19) are identical to formulas (7) and (8), respectively, in [255], [267, p. 55]. Ramanujan's discourses in [255] and in the present manuscript indicate that Ramanujan was regarding the three self-reciprocal Fourier transforms in [255] and the five self-reciprocal Fourier transforms here as known. The identities (18.2.18), (18.2.20), and (18.2.21) are especially easy to prove, because each can be established by expanding the denominator in a geometric series and integrating termwise. The identities (18.2.19) and (18.2.23) are more difficult to prove. One can find (18.2.18), (18.2.20), and (18.2.21) in the *Tables* [126, p. 537, formula 3.981, no. 3; page 538, formula 3.981, no. 10; p. 537, formula 3.981, no. 6].

Formula (18.2.19) is a special case of a more general integral evaluation given in [126, p. 539, formula 3.983, no. 6], where the requirement $a < 0$ given there is spurious. In that formula, set $a = 3n$, $b = \frac{1}{3}\pi$, $\beta = 0$, and $\gamma = \pi$ to deduce that

$$\begin{aligned}\int_0^{\infty} \frac{\cos 3nx}{2\cosh \pi x + 1} dx &= \frac{\cosh 4n - \cosh 2n}{\sqrt{3}(\cosh 6n - 1)} \\ &= \frac{2\cosh^2 2n - 1 - \cosh 2n}{\sqrt{3}(4\cosh^3 2n - 3\cosh 2n - 1)} \\ &= \frac{1}{\sqrt{3}(2\cosh 2n + 1)},\end{aligned}$$

because $4x^3 - 3x - 1 = (2x^2 - x - 1)(2x + 1)$.

Lastly, (18.2.22) is a special case of a more general integral formula found in [126, p. 539, formula 3.984, no. 3], where the factor π is unfortunately missing in the evaluation, i.e., the formula should read

$$\int_0^{\infty} \frac{\sin ax \sinh \frac{1}{2}x}{\cosh x + \cos \beta} dx = \pi \frac{\sinh(a\beta)}{2\sin \frac{1}{2}\beta \cosh(a\pi)}. \tag{18.3.1}$$

It will be necessary to make a change of variable $x = 2ru$, $r > 0$, to rewrite (18.3.1) as

$$\int_0^\infty \frac{\sin(2aru)\sinh(ru)}{\cosh(2ru)+\cos\beta}du = \pi\frac{\sinh(a\beta)}{4r\sin\frac{1}{2}\beta\,\cosh(a\pi)}. \tag{18.3.2}$$

Setting $r = \pi$, $a = 3n/\pi$, and $\beta = \frac{2}{3}\pi$ in (18.3.2), we deduce that

$$\begin{aligned}\int_0^\infty \frac{\sin(6nu)\sinh(\pi u)}{2\cosh(2\pi u)-1}du &= \frac{\sinh(2n)}{4\sqrt{3}\,\cosh(3n)}\\ &= \frac{\sinh n\cosh n}{2\sqrt{3}(4\cosh^3 n - 3\cosh n)}\\ &= \frac{\sinh n}{2\sqrt{3}(4\cosh^2 n - 3)}\\ &= \frac{\sinh n}{2\sqrt{3}(2\cosh 2n - 1)},\end{aligned}$$

which completes the proof of (18.2.22).

The beautiful identity (18.2.23) was first submitted as a problem to the *Journal of the Indian Mathematical Society* [245], [267, pp. 324–325], with Ramanujan's solution being one of the few published solutions by Ramanujan to his own problems. The same solution is sketched in Ramanujan's paper [255], [267, p. 55]. It is also given in Chap. 13 of his second notebook [268], [38, p. 225], where (in the latter source) the condition $\alpha\beta = \pi$ was unfathomably replaced by $\alpha\beta = \pi/4$. A. Dixit [111] has found an elegant extension of the following formula, also due to Ramanujan [257, Eq. (13)], which we provide below under a slight renaming of the parameters α and β. If α and β are two positive numbers such that $\alpha\beta = 1$, then

$$\begin{aligned}&\alpha^{-1/2} - 4\pi\alpha^{-3/2}\int_0^\infty \frac{xe^{-\pi x^2/\alpha^2}}{e^{2\pi x}-1}\,dx = \beta^{-1/2} - 4\pi\beta^{-3/2}\int_0^\infty \frac{xe^{-\pi x^2/\beta^2}}{e^{2\pi x}-1}\,dx\\ &= \frac{1}{4\pi\sqrt{\pi}}\int_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt,\end{aligned} \tag{18.3.3}$$

where $\Xi(x)$ denotes Riemann's Ξ-function. Observe that if the far left-hand side of (18.3.3) is shown to be equal to the far right-hand side, then the first equality follows readily from the relation $\alpha\beta = 1$.

The identity (18.2.24) is also stated by Ramanujan in [255], where a proof is sketched. Since (18.2.25)–(18.2.27) are not given by Ramanujan in [255], we provide proofs. The proofs are dependent on the elementary identities

$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/(4a^2)}, \tag{18.3.4}$$

where $\operatorname{Re} a > 0$ and b is real, and

$$\int_0^\infty xe^{-a^2x^2} \sin bx\, dx = \frac{b\sqrt{\pi}}{4a^3} e^{-b^2/(4a^2)}, \tag{18.3.5}$$

where $\operatorname{Re} a > 0$ and b is real [126, p. 515, formula 3.896, no. 4; p. 529, formula 3.952, no. 1].

Proof of (18.2.25). Using (18.2.20) and (18.3.4), we find that

$$\begin{aligned}\sqrt{\alpha}\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} e^{-(\alpha x)^2} dx &= 2\sqrt{2\alpha}\int_0^\infty e^{-\alpha^2x^2} dx \int_0^\infty \frac{\cosh \pi u}{\cosh 2\pi u} \cos(4\pi x u) du \\ &= 2\sqrt{2\alpha}\int_0^\infty \frac{\cosh \pi u}{\cosh 2\pi u} du \int_0^\infty e^{-\alpha^2x^2} \cos(4\pi x u) dx \\ &= \sqrt{\frac{2\pi}{\alpha}}\int_0^\infty \frac{\cosh \pi u}{\cosh 2\pi u} e^{-4\pi^2u^2/\alpha^2} du \\ &= \sqrt{\beta}\int_0^\infty \frac{\cosh \pi u}{\cosh 2\pi u} e^{-(\beta u)^2} du,\end{aligned}$$

since $\alpha\beta = 2\pi$. □

Proof of (18.2.26). Using (18.2.21) and (18.3.5), we see that

$$\begin{aligned}&\sqrt{\alpha}\int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x} \alpha x e^{-(\alpha x)^2} dx \\ &\quad = 2\alpha\sqrt{2\alpha}\int_0^\infty xe^{-\alpha^2x^2} dx \int_0^\infty \frac{\sinh \pi u}{\cosh 2\pi u} \sin(4\pi x u) du \\ &\quad = 2\alpha\sqrt{2\alpha}\int_0^\infty \frac{\sinh \pi u}{\cosh 2\pi u} du \int_0^\infty xe^{-\alpha^2x^2} \sin(4\pi x u) dx \\ &\quad = \left(\frac{2\pi}{\alpha}\right)^{3/2}\int_0^\infty \frac{\sinh \pi u}{\cosh 2\pi u} u e^{-4\pi^2u^2/\alpha^2} du \\ &\quad = \sqrt{\beta}\int_0^\infty \frac{\sinh \pi u}{\cosh 2\pi u} \beta u e^{-(\beta u)^2} du,\end{aligned}$$

because $\alpha\beta = 2\pi$. □

Proof of (18.2.27). Employing (18.2.22) and (18.3.5), we readily find that

$$\begin{aligned}&\sqrt{\alpha}\int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1} \alpha x e^{-(\alpha x)^2} dx \\ &\quad = 2\sqrt{3}\,\alpha^{3/2}\int_0^\infty xe^{-\alpha^2x^2} dx \int_0^\infty \frac{\sinh \pi u}{2\cosh 2\pi u - 1} \sin(6\pi x u) du\end{aligned}$$

$$= 2\sqrt{3}\,\alpha^{3/2}\int_0^\infty \frac{\sinh \pi u}{2\cosh 2\pi u - 1}du \int_0^\infty xe^{-\alpha^2x^2}\sin(6\pi xu)dx$$
$$= \left(\frac{3\pi}{\alpha}\right)^{3/2}\int_0^\infty \frac{\sinh \pi u}{2\cosh 2\pi u - 1}ue^{-9\pi^2u^2/\alpha^2}du$$
$$= \sqrt{\beta}\int_0^\infty \frac{\sinh \pi u}{2\cosh 2\pi u - 1}\beta ue^{-(\beta u)^2}du,$$

since $\alpha\beta = 3\pi$. ☐

It is difficult to ascertain Ramanujan's intention in the last paragraph of the manuscript, since the content of §§2–3 is unknown to us. However, the five L-functions listed by Ramanujan are associated with the five self-reciprocal functions (18.2.18)–(18.2.22), respectively. For these connections, see E.C. Titchmarsh's text [305, pp. 262–263]. Titchmarsh does not divulge who first established the self-reciprocal relations (18.2.18)–(18.2.22). Is it possible that the contents of this partial manuscript were communicated by Hardy to Titchmarsh, who was his former doctoral student?

18.4 Page 192

It is not clear that page 192 is a portion of the previous manuscript, but the editor's decision to place this page after pages 190 and 191 is reasonable. The three entries on this page are labeled 28, 32, and 36, with a fourth labeled 40 and empty. The last entry on page 191 is labeled 27), but note that a different tagging notation is used. We quote the three entries.

28. $1^{-s} + 3^{-s}\omega\varpi + 5^{-s}\omega^5 + 9^{-s}\omega^2 + 11^{-s}\omega^4\varpi + 13^{-s}\omega^3 + 15^{-s}\varpi$
$+ 17^{-s}\omega + 19^{-s}\omega^5\varpi + 23^{-s}\omega^2\varpi + 25^{-s}\omega^4 + 27^{-s}\omega^3\varpi + \cdots$
where $\omega^6 = \varpi^2 = 1$,

32. $1^{-s} + 3^{-s}\omega + 5^{-s}\omega^3\varpi + 7^{-s}\omega^6\varpi + 9^{-s}\omega^2 + 11^{-s}\omega^7 + 13^{-s}\omega^5\varpi$
$+ 15^{-s}\omega^4\varpi + 17^{-s}\omega^4 + 19^{-s}\omega^5 + 21^{-s}\omega^7\varpi + 23^{-s}\omega^2\varpi + 25^{-s}\omega^6$
$+ 27^{-s}\omega^3 + 29^{-s}\omega\varpi + 31^{-s}\varpi + \cdots$
where $\omega^8 = \varpi^2 = 1$,

36. $1^{-s} + 5^{-s}\omega + 7^{-s}\omega^2\varpi + 11^{-s}\omega^5\varpi + 13^{-s}\omega^4 + 17^{-s}\omega^3 + 19^{-s}\varpi$
$+ 23^{-s}\omega\varpi + 25^{-s}\omega^2 + 29^{-s}\omega^5 + 31^{-s}\omega^4\varpi + 35^{-s}\omega^3\varpi + \cdots$
where $\omega^6 = \varpi^2 = 1$.

Note that Ramanujan has made no claims about these three series. Each is a Dirichlet L-function, with the periods of the characters being 28, 32, and 36, respectively.

18.5 Explicit Evaluations of Certain Quotients of *L*-Series

Let

$$\chi_4(n) = \begin{cases} (-1)^m, & \text{if } n = 2m+1, \\ 0, & \text{if } n \text{ is even,} \end{cases}$$

and let $\chi_3(n)$ denote the Legendre symbol $\left(\frac{n}{3}\right)$.

Entry 18.5.1 (p. 203). *Let $\zeta(s)$ denote the Riemann zeta function, and let $L(s,\chi_4)$ and $L(s,\chi_3)$ denote the Dirichlet L-functions associated with the characters χ_4 and χ_3 defined above. Then*

$$\frac{(1-2^{2/3})\zeta(1/3)}{(1-2^{1/3})\zeta(2/3)} = \frac{\pi^{1/3}}{\Gamma(1/3)} \frac{2^{1/3}+4^{1/3}}{\sqrt{3}}, \tag{18.5.1}$$

$$\frac{L(1/3,\chi_4)}{L(2/3,\chi_4)} = \frac{\pi^{1/3}}{\Gamma(1/3)} \cdot 4^{1/3}, \tag{18.5.2}$$

$$\frac{L(1/3,\chi_3)}{L(2/3,\chi_3)} = \frac{\pi^{1/3}}{\Gamma(1/3)} \cdot 2^{1/3} \cdot 3^{1/6}, \tag{18.5.3}$$

$$\frac{(1-2^{3/4})\zeta(1/4)}{(1-2^{1/4})\zeta(3/4)} = \frac{\pi^{1/4}}{\Gamma(1/4)} \sqrt{3+2^{5/4}}, \tag{18.5.4}$$

$$\frac{L(1/4,\chi_4)}{L(3/4,\chi_4)} = \frac{\pi^{1/4}}{\Gamma(1/4)} \sqrt{2+2\sqrt{2}}, \tag{18.5.5}$$

$$\frac{L(1/4,\chi_3)}{L(3/4,\chi_3)} = \frac{\pi^{1/4}}{\Gamma(1/4)} \sqrt{\sqrt{3}+\sqrt{6}}. \tag{18.5.6}$$

Proof. We recall the functional equation of $\zeta(s)$ [306, p. 22],

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-(1-s)/2}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s). \tag{18.5.7}$$

We also need the functional equations of the two L-functions cited above [101, p. 71, Eq. (11)]

$$\pi^{-(2-s)/2}4^{(2-s)/2}\Gamma(1-\tfrac{1}{2}s)L(1-s,\chi_4) = \pi^{-(s+1)/2}4^{(s+1)/2}\Gamma(\tfrac{1}{2}(s+1))L(s,\chi_4) \tag{18.5.8}$$

and

$$\begin{aligned}\pi^{-(2-s)/2}3^{(2-s)/2}&\Gamma(1-\tfrac{1}{2}s)L(1-s,\chi_3)\\ &= \pi^{-(s+1)/2}3^{(s+1)/2}\Gamma(\tfrac{1}{2}(s+1))L(s,\chi_3).\end{aligned} \tag{18.5.9}$$

The duplication formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x+\tfrac{1}{2}) \tag{18.5.10}$$

and the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{18.5.11}$$

are also needed in our calculations.

We first prove (18.5.1). Setting $s = \frac{1}{3}$ in (18.5.7), we find that

$$\frac{(1-2^{2/3})\zeta(1/3)}{(1-2^{1/3})\zeta(2/3)} = \frac{(1+2^{1/3})\zeta(1/3)}{\zeta(2/3)} = (1+2^{1/3})\pi^{-1/6}\frac{\Gamma(1/3)}{\Gamma(1/6)}. \tag{18.5.12}$$

Setting $x = \frac{1}{6}$ in (18.5.10), we readily find that

$$\frac{\Gamma(1/3)}{\Gamma(1/6)} = \frac{2^{-2/3}}{\sqrt{\pi}}\Gamma(2/3). \tag{18.5.13}$$

Using (18.5.13) in (18.5.12), we deduce that

$$\begin{aligned}\frac{(1-2^{2/3})\zeta(1/3)}{(1-2^{1/3})\zeta(2/3)} &= (1+2^{1/3})(2\pi)^{-2/3}\Gamma(2/3) \\ &= \frac{\pi^{1/3}}{\sqrt{3}\Gamma(1/3)}(2^{1/3}+2^{2/3}),\end{aligned}$$

upon the use of the reflection formula (18.5.11) with $x = \frac{2}{3}$. This then completes the proof of (18.5.1).

The proof of (18.5.2) is next on our agenda. Setting $s = \frac{2}{3}$ in (18.5.8), we easily arrive at

$$\frac{L(1/3,\chi_4)}{L(2/3,\chi_4)} = \pi^{-1/6}2^{1/3}\frac{\Gamma(5/6)}{\Gamma(2/3)} = \frac{2^{4/3}\pi^{5/6}}{\Gamma(2/3)\Gamma(1/6)},$$

by the reflection formula (18.5.11). Using next the duplication formula (18.5.10), we see that

$$\frac{L(1/3,\chi_4)}{L(2/3,\chi_4)} = \frac{\pi^{5/6}2^{4/3}}{\Gamma(2/3)}\frac{2^{-2/3}}{\sqrt{\pi}}\frac{\Gamma(2/3)}{\Gamma(1/3)} = \frac{\pi^{1/3}4^{1/3}}{\Gamma(1/3)},$$

and this completes the proof of (18.5.2).

We establish (18.5.3). In (18.5.9), we set $s = \frac{2}{3}$ and use the reflection and duplication formulas (18.5.11) and (18.5.10) to find that

$$\frac{L(1/3,\chi_3)}{L(2/3,\chi_3)} = \pi^{-1/6}3^{1/6}\frac{\Gamma(5/6)}{\Gamma(2/3)} = \frac{2\pi^{5/6}3^{1/6}}{\Gamma(2/3)\Gamma(1/6)} = \frac{\pi^{1/3}3^{1/6}2^{1/3}}{\Gamma(1/3)}.$$

To prove (18.5.4), we set $s = \frac{1}{4}$ in the functional equation (18.5.7), employ the reflection and duplication formulas (18.5.11) and (18.5.10), and lastly recall the value

$$\sin^2\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2}+1}{2\sqrt{2}}.$$

Hence,

$$\frac{(1-2^{3/4})\zeta(1/4)}{(1-2^{1/4})\zeta(3/4)} = (1+2^{1/4}+\sqrt{2})\frac{\zeta(1/4)}{\zeta(3/4)} = \frac{(1+2^{1/4}+\sqrt{2})\pi^{1/4}}{\sqrt{\sqrt{2}+1}\,\Gamma(1/4)}.$$

Since

$$\frac{(1+2^{1/4}+\sqrt{2})}{\sqrt{\sqrt{2}+1}} = \sqrt{3+2^{5/4}},$$

we see that the proof of (18.5.4) is complete.

The proof of (18.5.5) follows along the same lines. We set $s = \frac{3}{4}$ in (18.5.8) and use the reflection and duplication formulas, (18.5.11) and (18.5.10), to deduce that

$$\frac{L(1/4,\chi_4)}{L(3/4,\chi_4)} = \frac{\pi^{1/4}}{\Gamma(1/4)2^{1/4}\sin(\pi/8)}.$$

If we now use the facts

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

and

$$\sqrt{2+2\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{\sqrt{2}-1}},$$

we complete the proof of (18.5.5).

Lastly, put $s = \frac{3}{4}$ in the functional equation (18.5.9) and proceed in exactly the same manner as in the previous proof to arrive at

$$\frac{L(1/4,\chi_3)}{L(3/4,\chi_3)} = \frac{\pi^{1/4}3^{1/4}}{\Gamma(1/4)\sqrt{\sqrt{2}-1}}.$$

If we now use the identity

$$\frac{3^{1/4}}{\sqrt{\sqrt{2}-1}} = \sqrt{\sqrt{3}+\sqrt{6}},$$

we finish the proof of (18.5.6). □

19

Miscellaneous Results in Analysis

19.1 Introduction

Recall that when Ramanujan's lost notebook [269] was published in 1988, other fragments and partial manuscripts were also published with the lost notebook. In the first portion of this chapter, we examine two formulas found on page 336 of [269] that are clearly wrong. Undoubtedly, Ramanujan realized that these results are indeed incorrect as they stand. He possibly possessed correct identities and used some unknown formal procedure to replace certain expressions by divergent series in order to make the identities more attractive. Ramanujan frequently enjoyed stating identities in an unorthodox fashion in order to surprise or titillate his audience. We timorously conjecture that Ramanujan had established correct identities in each case, but we do not know what they are.

Following our discussion of these two intriguing but incorrect formulas, we consider various isolated results. Perhaps the most interesting are an integral-series identity on page 197 and a study of the integral $\int_0^x \frac{\sin u}{u} du$, for which Ramanujan determines the points where it achieves local maxima and minima.

19.2 Two False Claims

Entry 19.2.1 (p. 336). *Let* $\sigma_s(n) = \sum_{d|n} d^s$, *and let* $\zeta(s)$ *denote the Riemann zeta function. Then*

$$\Gamma(s+\tfrac{1}{2})\left\{\frac{\zeta(1-s)}{(s-\frac{1}{2})x^{s-\frac{1}{2}}} + \frac{\zeta(-s)\tan\frac{1}{2}\pi s}{2x^{s+\frac{1}{2}}} + \sum_{n=1}^{\infty}\frac{\sigma_s(n)}{2i}\left\{(x-in)^{-s-\frac{1}{2}} - (x+in)^{-s-\frac{1}{2}}\right\}\right\}$$

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$$= (2\pi)^s \left\{ \frac{\zeta(1-s)}{2\sqrt{\pi x}} - 2\pi\sqrt{\pi x}\zeta(-s)\tan\tfrac{1}{2}\pi s \right.$$
$$\left. + \sqrt{\pi}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\right\}. \tag{19.2.1}$$

Entry 19.2.2 (p. 336). *Let $\sigma_s(n)$ and $\zeta(s)$ be as in the preceding entry. If α and β are positive numbers such that $\alpha\beta = 4\pi^2$, then*

$$\alpha^{(s+1)/2}\left\{\frac{1}{\alpha}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\tfrac{1}{2}\pi s+\sum_{n=1}^{\infty}\sigma_s(n)\sin(n\alpha)\right\}$$
$$= \beta^{(s+1)/2}\left\{\frac{1}{\beta}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\tfrac{1}{2}\pi s+\sum_{n=1}^{\infty}\sigma_s(n)\sin(n\beta)\right\}. \tag{19.2.2}$$

Each of Ramanujan's claims is easily seen to be false in general, because each contains divergent series. In Sects. 19.3–19.6, we examine these two formulas. Formula (19.2.2) is especially intriguing because of its beautiful symmetry, because it appears to be a relation between Eisenstein series formally extended to the real line, and because it appears to be an analogue of the Poisson summation formula or a special instance of the Voronoï summation formula.

19.3 First Attempt: A Possible Connection with Eisenstein Series

A first examination of (19.2.2) reminds us of the transformation formulas for Eisenstein series when s is a positive odd integer. In [29], Berndt derived modular transformation formulas for a large class of analytic Eisenstein series. Specializing Theorem 2 of [29] for $r_1 = r_2 = 0$ and the modular transformation $Tz = -1/z$, for $z \in \mathcal{H} = \{z : \operatorname{Im} z > 0\}$ we find that for any complex number s,

$$z^{-s}(1+e^{\pi is})\sum_{n=1}^{\infty}\sigma_{s-1}(n)e^{-2\pi in/z} = (1+e^{\pi is})\sum_{n=1}^{\infty}\sigma_{s-1}(n)e^{2\pi inz}$$
$$- z^{-s}e^{\pi is}(2\pi i)^{-s}(1+e^{\pi is})\Gamma(s)\zeta(s) + (2\pi i)^{-s}(1+e^{\pi is})\Gamma(s)\zeta(s)$$
$$- (2\pi i)^{-s}\int_C u^{s-1}\frac{1}{e^{zu}-1}\frac{1}{e^u-1}du, \tag{19.3.1}$$

where $\zeta(s)$ denotes the Riemann zeta function. Here C is a loop beginning at $+\infty$, proceeding to the left in $\mathcal{H}$, encircling the origin in the positive direction so that $u = 0$ is the only zero of $(e^{zu}-1)(e^u-1)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. We choose the branch of u^s with $0 < \arg u < 2\pi$. Otherwise, outside the integrand, we choose the branch of $\log w$ such that $-\pi \leq \arg w < \pi$. Replacing s by $s+1$ in (19.3.1) and slightly simplifying, we find that

$$\begin{aligned} z^{-s-1}\sum_{n=1}^{\infty}\sigma_s(n)e^{-2\pi in/z} &= \sum_{n=1}^{\infty}\sigma_s(n)e^{2\pi inz} \\ &+ z^{-s-1}e^{\pi is}(2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) + (2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) \\ &- \frac{(2\pi i)^{-s-1}}{1-e^{\pi is}}\int_C u^s \frac{1}{e^{zu}-1}\frac{1}{e^u-1}du. \end{aligned} \tag{19.3.2}$$

Next, recall the functional equation of the Riemann zeta function (3.1.4) [306, p. 16, equation (2.1.8)],

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\tfrac{1}{2}\pi s\right)\Gamma(s)\zeta(s). \tag{19.3.3}$$

If we replace s by $s+1$ in (19.3.3), we easily see that

$$(2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) = \frac{ie^{-\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)}. \tag{19.3.4}$$

Using (19.3.4) in (19.3.2), we conclude that

$$\begin{aligned} z^{-s-1}\sum_{n=1}^{\infty}\sigma_s(n)e^{-2\pi in/z} &= \sum_{n=1}^{\infty}\sigma_s(n)e^{2\pi inz} + z^{-s-1}\frac{ie^{\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)} \\ &+ \frac{ie^{-\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)} - \frac{(2\pi i)^{-s-1}}{1-e^{\pi is}}\int_C u^s\frac{1}{e^{zu}-1}\frac{1}{e^u-1}du. \end{aligned} \tag{19.3.5}$$

Omitting n, note that the product of the arguments in the exponentials in the two infinite series in (19.3.5) is equal to $4\pi^2$, in accordance with the condition $\alpha\beta = 4\pi^2$ prescribed by Ramanujan. Equation (19.3.5) is as close to (19.2.2) as we can get using the chief theorem from [29].

19.4 Second Attempt: A Formula in Ramanujan's Paper [257]

We conjecture that Ramanujan's formula (19.2.2) arose from the research that produced his paper [257], [267, pp. 72–77]. On page 75 in [267], in formula (15), Ramanujan asserts that if $\operatorname{Re} s > -1$ and if α and β are positive numbers such that $\alpha\beta = 4\pi^2$, then

$$\begin{aligned} &\frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)}\alpha^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)}\alpha^{(s+1)/2} \\ &\qquad + \alpha^{(s+1)/2}\int_0^{\infty}\int_0^{\infty}\frac{x^s\sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy \end{aligned}$$

$$= \frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)}\beta^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)}\beta^{(s+1)/2}$$
$$+ \beta^{(s+1)/2}\int_0^\infty \int_0^\infty \frac{x^s \sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy. \qquad (19.4.1)$$

Suppose that we multiply both sides of (19.4.1) by $4\cos(\frac{1}{2}\pi s)$ to deduce that

$$\alpha^{(s+1)/2}\left\{\frac{1}{\alpha}\zeta(1-s) + \frac{1}{2}\zeta(-s)\cot(\tfrac{1}{2}\pi s) + 4\cos(\tfrac{1}{2}\pi s)\int_0^\infty\int_0^\infty \frac{x^s\sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy\right\}$$
$$=\beta^{(s+1)/2}\left\{\frac{1}{\beta}\zeta(1-s) + \frac{1}{2}\zeta(-s)\cot(\tfrac{1}{2}\pi s) + 4\cos(\tfrac{1}{2}\pi s)\int_0^\infty\int_0^\infty \frac{x^s\sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy\right\}. \qquad (19.4.2)$$

We note that the first two expressions on each side of (19.4.2) are identical to the first two terms on each side of (19.2.2), except that $\tan(\frac{1}{2}\pi s)$ in (19.2.2) has been replaced by $\cot(\frac{1}{2}\pi s)$ in (19.4.2). However, we are unable to make any identification of the double integrals in (19.4.2) with the divergent sums in (19.2.2).

19.5 Third Attempt: The Voronoï Summation Formula

Our third attempt to prove Entries 19.2.2 and 19.2.1 depends on the Voronoï summation formula. We only briefly sketch the background and hypotheses needed for the statement of the Voronoï summation formula. For complete details, see the papers [26–28], and [89].

Let $s = \sigma + it$, with σ and t real, and let

$$\phi(s) := \sum_{n=1}^\infty a(n)\lambda_n^{-s} \qquad \text{and} \qquad \psi(s) := \sum_{n=1}^\infty b(n)\mu_n^{-s}, \qquad 0 < \lambda_n, \mu_n \to \infty,$$

be two Dirichlet series with abscissas of absolute convergence σ_a and σ_a^*, respectively. Let $r > 0$, and suppose that $\phi(s)$ and $\psi(s)$ satisfy a functional equation of the type

$$\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s). \qquad (19.5.1)$$

Define also

$$Q(x) := \frac{1}{2\pi i}\int_C \frac{\phi(s)x^s}{s}ds, \qquad (19.5.2)$$

where C is a simple closed curve(s) containing the integrand's poles in its interior.

The Voronoï summation formula in its original form with $a(n) = d(n)$, where $d(n)$ denotes the number of positive divisors of the positive integer n, was first proved by M.G. Voronoï in 1904 [310]. Since then, "Voronoï summation formulas" have been established for a variety of arithmetic functions under various hypotheses. In particular, Berndt [28] established various versions of the Voronoï summation formula, including the following theorem from [28, p. 142, Theorem 1], where several references to the literature on Voronoï summation formulas can be found.

Theorem 19.5.1. *Let $f \in C^{(1)}(0, \infty)$. Then, if $0 < a < \lambda_1 < x < \infty$,*

$$\sum_{\lambda_n \le x}{}' a(n) f(\lambda_n) = \int_a^x Q'(t) f(t) dt + \sum_{n=1}^{\infty} b(n) \int_a^x \left(\frac{t}{\mu_n}\right)^{(r-1)/2} J_{r-1}(2\sqrt{\mu_n t}) f(t) dt, \tag{19.5.3}$$

where the prime $'$ on the summation sign on the left-hand side indicates that if $x = \lambda_n$, for some integer n, then only $\frac{1}{2}a(n)f(\lambda_n)$ is counted, and where $J_\nu(x)$ denotes the ordinary Bessel function of order ν.

This is the simplest theorem of this sort. The two applications that we make of Theorem 19.5.1 are formal in the sense that there are no versions of the Voronoï summation formula that would ensure the validity of our applications; indeed, as we remarked above, both (19.2.1) and (19.2.2) contain divergent series. Possibly Ramanujan discovered some version of the Voronoï summation formula for $a(n) = \sigma_k(n)$, but if so, he apparently had established neither a precise version nor conditions for its validity. Under this assumption, we next see how Ramanujan *might* have been led to the two entries above.

In order to avoid possible confusion, we are going to replace s by k in our attempts to prove (19.2.1) and (19.2.2). It is well known and easy to prove that for any real number k,

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}, \qquad \sigma > \sup\{1, k+1\}. \tag{19.5.4}$$

Then with the use of the functional equation (19.3.3) for $\zeta(s)$, it is not difficult to show that if k is an odd integer [89, p. 17],

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(k-s) = (-1)^{(k+1)/2}(2\pi)^{-(k+1-s)}\Gamma(k+1-s)\zeta(k+1-s)\zeta(1-s). \tag{19.5.5}$$

Thus, in the settings (19.5.1) and (19.5.5), we have

$$a(n) = \sigma_k(n), \qquad b(n) = (-1)^{(k+1)/2}\sigma_k(n), \qquad k \text{ odd}, \tag{19.5.6}$$

$$\lambda_n = \mu_n = 2\pi n, \qquad n \geq 1, \qquad r = k+1. \tag{19.5.7}$$

Furthermore, $Q(x)$ is the sum of the residues of

$$\frac{(2\pi)^{-s}\zeta(s)\zeta(s-k)x^s}{s}$$

taken over all its poles, which are at $s=1$, $s=k+1$, and $s=0$. Since $\zeta(s)$ has a simple pole at $s=1$ with residue 1 and [306, p. 19]

$$\zeta(0) = -\frac{1}{2}, \tag{19.5.8}$$

we find that

$$Q(x) = -\frac{1}{2}\zeta(-k) + \frac{\zeta(1-k)x}{2\pi} + \frac{\zeta(k+1)x^{k+1}}{(2\pi)^{k+1}(k+1)}.$$

It follows that

$$Q'(x) = \frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)x^k}{(2\pi)^{k+1}}. \tag{19.5.9}$$

We first examine (19.2.2). In our formal application of (19.5.3), we clearly should set $a=0$, $x=\infty$, and $f(t)=\sin(\alpha t/(2\pi))$. In order to apply (19.5.3), we need to employ the integral evaluation [126, p. 773, formula 6.728, no. 5]

$$\int_0^\infty x^{k+1}J_k(bx)\sin(ax^2)dx = \frac{b^k}{(2a)^{k+1}}\cos\left(\frac{b^2}{4a} - \frac{k\pi}{2}\right). \tag{19.5.10}$$

Hence, using (19.5.10), we find that

$$\begin{aligned}\int_0^\infty t^{k/2}J_k(2\sqrt{2\pi nt})\sin\left(\frac{\alpha t}{2\pi}\right)dt &= 2\int_0^\infty u^{k+1}J_k(2\sqrt{\mu_n}\,u)\sin\left(\frac{\alpha u^2}{2\pi}\right)du\\ &= \frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\cos\left(\frac{4\pi^2 n}{\alpha} - \frac{k\pi}{2}\right)\\ &= (-1)^{(k-1)/2}\frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\sin\left(\frac{4\pi^2 n}{\alpha}\right)\\ &= (-1)^{(k-1)/2}\frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\sin(\beta n),\end{aligned} \tag{19.5.11}$$

since $\alpha\beta = 4\pi^2$.

With the preliminary details out of the way, we are now ready to apply the Voronoï summation formula (19.5.3). Using the calculations (19.5.9) and (19.5.11) and the parameters defined above, we formally find that

$$\sum_{n=1}^{\infty} \sigma_k(n) \sin(\alpha n) = \int_0^{\infty} \left(\frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)t^k}{(2\pi)^{k+1}} \right) \sin\left(\frac{\alpha t}{2\pi}\right) dt$$
$$- \left(\frac{2\pi}{\alpha}\right)^{k+1} \sum_{n=1}^{\infty} \sigma_k(n) \sin(\beta n). \tag{19.5.12}$$

Thus, if we replace s by k in (19.2.2) and assume that k is an odd integer, then (19.5.12) is as close as we can get in our efforts to formally derive (19.2.2). Note that on the right side of (19.5.12) a minus sign appears, in contrast to the right side of (19.2.2), and that a divergent integral appears on the right-hand side of (19.5.12) in place of the expressions involving the Riemann zeta function appearing in (19.2.2).

We now turn to (19.2.1). Observe that the infinite series on the left-hand side are reminiscent of the *finite* Riesz sums $\sum_{n\leq x} \sigma_s(n)(x-n)^r$, for which identities have been derived by, for example, A. Oppenheim [239] and A. Laurinčikas [209]. Once more, we make an application of the Voronoï summation formula. Note that the series on the left-hand side of (19.2.1) does not converge for any real value of s, since $\sigma_s(n) \geq n^s$. Also note that for x sufficiently large and for $\sigma > \frac{1}{2}$, each expression in (19.2.1) tends to 0 as x tends to ∞, except for $-2\pi\sqrt{\pi x}\zeta(-s)\tan\frac{1}{2}\pi s$, which tends to ∞.

To effect our application of Theorem 19.5.1, we need the integral evaluation [126, p. 709, formula 6.565, no. 2]

$$\int_0^{\infty} x^{\nu+1} J_\nu(bx)(x^2+a^2)^{-\nu-1/2} dx = \frac{\sqrt{\pi}\, b^{\nu-1}}{2e^{ab}\Gamma(\nu+\frac{1}{2})}, \tag{19.5.13}$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re} \nu > -\frac{1}{2}$, and $J_\nu(x)$ denotes the ordinary Bessel function of order ν. Apply the Voronoï summation formula (19.5.3) twice, with $a = 0$, $x = \infty$, and $f(t) = (x \mp it)^{-k-1/2}$, under the same conditions (19.5.6) and (19.5.7) as in our previous application. We do not provide further details but invite readers to consult the paper by Berndt, O.-Y. Chan, S.-G. Lim, and A. Zaharescu [48], where the remainder of the failed proof can be found. We eventually then arrive at the "identity"

$$\sum_{n=1}^{\infty} \sigma_k(n) \left\{ (x-in)^{-k-1/2} - (x+in)^{-k-1/2} \right\}$$
$$= \int_0^{\infty} \left(\frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)t^k}{(2\pi)^{k+1}} \right) \left((x-it)^{-k-1/2} - (x+it)^{-k-1/2} \right) dt$$
$$- \frac{i\sqrt{2}}{\Gamma(k+\frac{1}{2})} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{\sqrt{n}} e^{-2\sqrt{\pi n x}} \sin\left(2\sqrt{\pi n x} + \frac{1}{4}\pi\right), \tag{19.5.14}$$

which should be compared with (19.2.1). Observe that the integral on the right-hand side of (19.5.14) diverges, although it can be subdivided into two improper integrals, one of which converges and is elementary, and the other of which diverges.

19.6 Fourth Attempt: Mellin Transforms

Another effort to prove Entries 19.2.1 and 19.2.2 has utilized Mellin transforms. We refer readers to the aforementioned paper by Berndt, Chan, Lim, and Zaharescu [48] for the details of this failed attempt.

19.7 An Integral on Page 197

Entry 19.7.1 (p. 197). *Let $n > 0$ and let $t > 0$. Then*

$$\int_0^\infty \frac{\sin(\pi t x)}{x\cosh(\pi x)} e^{-i\pi n x^2} dx = \frac{\pi}{2} - 2\sum_{k=0}^\infty (-1)^k \frac{e^{-(2k+1)\pi t/2 + (2k+1)^2 i\pi n/4}}{2k+1} - \frac{\pi}{\sqrt{n}} e^{-i\pi/4} \int_0^\infty \sum_{k=0}^\infty (-1)^k e^{(t+u+(2k+1)i)^2 i\pi/(4n)} du. \quad (19.7.1)$$

Ramanujan has a slight misprint in his formulation of (19.7.1) in [269]; he forgot the factor π in the exponents in the summands in the first series on the right-hand side.

Before proving Entry 19.7.1, we state the values of some integrals that we need in our proof. For $a, b > 0$ [126, p. 542, formulas 3.989, nos. 5, 6],

$$\int_0^\infty \frac{\sin(\pi a x^2)\cos(bx)}{\cosh(\pi x)} dx = -\sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/2} \sin\left(\frac{(2k+1)^2\pi a}{4}\right) + \frac{1}{\sqrt{a}} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/(2a)} \sin\left(\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(2k+1)^2\pi}{4a}\right) \quad (19.7.2)$$

and

$$\int_0^\infty \frac{\cos(\pi a x^2)\cos(bx)}{\cosh(\pi x)} dx = \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/2} \cos\left(\frac{(2k+1)^2\pi a}{4}\right) + \frac{1}{\sqrt{a}} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/(2a)} \cos\left(\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(2k+1)^2\pi}{4a}\right). \quad (19.7.3)$$

In [126], the factor $(-1)^k$ has unfortunately been omitted from both sums in (19.7.2) and from the latter sum in (19.7.3). These formulas, including the mistakes, were copied from the tables of integral transforms [115, p. 36]. Next, for $a > 0$ and $\operatorname{Re} b > 0$ [126, p. 545, formula 4.111, no. 7],

$$\int_0^\infty \frac{\sin(ax)}{x\cosh(bx)} dx = 2\tan^{-1}\left(\exp\frac{\pi a}{2b}\right) - \frac{\pi}{2}. \quad (19.7.4)$$

Proof of Entry 19.7.1. Our uninspiring method of proof is undoubtedly not that used by Ramanujan, because our proof is a verification. We show that the derivatives of both sides of (19.7.1) as functions of t are equal. We then show that the limits of both sides of (19.7.1) as $t \to \infty$ are both equal to $\pi/2$ to conclude the proof. To that end, let $F(t)$ and $G(t)$ denote the left- and right-hand sides of (19.7.1). Then, using (19.7.2) and (19.7.3) with $a = n$ and $b = \pi t$, we find that

$$\begin{aligned} F'(t) &= \pi \int_0^\infty \frac{\sin(\pi t x)}{\cosh(\pi x)} e^{-i\pi n x^2} dx \\ &= \pi \left(\sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2} e^{i(2k+1)^2 \pi n/4} \right. \qquad (19.7.5) \\ &\quad \left. + \frac{1}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k e^{-(2k=1)\pi t/n} \exp\left(-i \left(\frac{\pi}{4} - \frac{\pi t^2}{4n} + \frac{(2k+1)^2 \pi}{4n} \right)\right)\right). \end{aligned}$$

On the other hand, by easily justified differentiations under the summation and integral signs and an inversion in order of integration and summation by absolute convergence,

$$\begin{aligned} G'(t) &= \pi \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2 + (2k+1)^2 \pi i n/4} \\ &\quad - 2\frac{\pi i}{4n} \frac{\pi}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k \int_0^\infty (t+u+(2k+1)i) e^{(t+u+(2k+1)i)^2 i\pi/(4n)} du \\ &= \pi \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2 + (2k+1)^2 \pi i n/4} \\ &\quad + \frac{\pi}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k e^{(t+u+(2k+1)i)^2 i\pi/(4n)}. \qquad (19.7.6) \end{aligned}$$

A comparison of (19.7.5) and (19.7.6) shows that indeed $F'(t) = G'(t)$. So, it remains to show that $F(t)$ and $G(t)$ are equal for some value of t.

We let t tend to ∞ to deduce the desired equality. Because of absolute and uniform convergence with respect to t in a neighborhood of ∞, we can let $t \to \infty$ under the integral and summation signs on the right side of (19.7.1) and readily deduce that

$$\lim_{t\to\infty} G(t) = \frac{\pi}{2}. \qquad (19.7.7)$$

On the other hand, write, with the use of (19.7.4),

$$F(t) = \int_0^\infty \left(\frac{\sin(\pi t x)}{x \cosh(\pi x)} e^{-i\pi n x^2} - \frac{\sin(\pi t x)}{x\cosh(\pi x)} \right) dx + \int_0^\infty \frac{\sin(\pi t x)}{x \cosh(\pi x)} dx$$

$$= \int_0^\infty \sin(\pi t x) \left(\frac{1}{x\cosh(\pi x)} e^{-i\pi n x^2} - \frac{1}{x\cosh(\pi x)} \right) dx \\ + 2\tan^{-1}\left(e^{\pi t/2}\right) - \frac{\pi}{2}. \tag{19.7.8}$$

Clearly, the function

$$\frac{1}{x\cosh(\pi x)} e^{-i\pi n x^2} - \frac{1}{x\cosh(\pi x)}$$

is in $L(-\infty, \infty)$. Hence, by (19.7.8) and a standard theorem from the theory of Fourier integrals [305, p. 11],

$$\lim_{t\to\infty} F(t) = 0 + 2\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}. \tag{19.7.9}$$

Thus, we see from (19.7.7) and (19.7.9) that $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t)$, and so the proof is complete. □

19.8 On the Integral $\int_0^x \frac{\sin u}{u} du$

On page 256 in [269], Ramanujan obtains explicit representations for the values of the local maxima and minima of the integral

$$\mathcal{S}(x) := \int_0^x \frac{\sin u}{u} du, \tag{19.8.1}$$

when $x > 0$. The integral $\mathcal{S}(x)$ is intimately connected with the sine and cosine integrals defined for $x > 0$ by [126, p. 936, formulas 8.230, nos. 1,2]

$$\operatorname{si}(x) := -\int_x^\infty \frac{\sin t}{t} dt \qquad \text{and} \qquad \operatorname{ci}(x) := \int_x^\infty \frac{\cos t}{t} dt. \tag{19.8.2}$$

Ramanujan first defines r, $r > 0$, and θ, $0 < \theta < \frac{1}{2}\pi$, by

$$r\cos\theta := \int_0^\infty \frac{e^{-xt}}{1+t^2} dt \qquad \text{and} \qquad r\sin\theta := \int_0^\infty \frac{te^{-xt}}{1+t^2} dt, \tag{19.8.3}$$

where $x > 0$. His first claim is the following identity.

Entry 19.8.1 (p. 256). *If r is defined by (19.8.3), then*

$$r^2 = \int_0^\infty \frac{e^{-xt}}{t} \log(1+t^2) dt.$$

Proof. From [126, p. 359, formula 3.354, nos. 1,2],

$$\int_0^\infty \frac{e^{-xt}}{1+t^2} dt = \mathrm{ci}(x)\sin x - \mathrm{si}(x)\cos x \tag{19.8.4}$$

and

$$\int_0^\infty \frac{te^{-xt}}{1+t^2} dt = -\mathrm{ci}(x)\cos x - \mathrm{si}(x)\sin x, \tag{19.8.5}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are defined by (19.8.2). Using the definitions (19.8.3) in conjunction with the foregoing identities, we easily see that

$$\begin{aligned} r^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= \{\mathrm{ci}(x)\sin x - \mathrm{si}(x)\cos x\}^2 + \{-\mathrm{ci}(x)\cos x - \mathrm{si}(x)\sin x\}^2 \\ &= \mathrm{ci}^2(x) + \mathrm{si}^2(x) \\ &= \int_0^\infty \frac{e^{-xt}}{t}\log(1+t^2)dt, \end{aligned}$$

where we have used another integral evaluation from the *Tables* [126, p. 609, formula 4.366, no. 1]. This completes the proof. □

Entry 19.8.2 (p. 256). *If r and θ are defined by (19.8.3) and $x > 0$, then*

$$\int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - r\cos(x-\theta) \tag{19.8.6}$$

and

$$\int_0^x \frac{1-\cos u}{u} du = \gamma + \log x - r\sin(x-\theta), \tag{19.8.7}$$

where γ denotes Euler's constant.

Proof. Again using (19.8.3)–(19.8.5), we easily find that

$$\begin{aligned} r\cos(x-\theta) &= r\cos x\cos\theta + r\sin x\sin\theta \\ &= \cos x\{\mathrm{ci}(x)\sin x - \mathrm{si}(x)\cos x\} \\ &\quad + \sin x\{-\mathrm{ci}(x)\cos x - \mathrm{si}(x)\sin x\} \\ &= -\ \mathrm{si}(x). \end{aligned} \tag{19.8.8}$$

The result (19.8.6) now follows from the definition (19.8.2) of $\mathrm{si}(x)$.

Next, from [126, p. 936, formula 8.230, no. 2],

$$\int_0^x \frac{1-\cos u}{u} du = \gamma + \log x - \mathrm{ci}(x). \tag{19.8.9}$$

A comparison of (19.8.9) with (19.8.7) indicates that in order to prove (19.8.7), all we need to do is to show that

$$r\sin(x-\theta)=\operatorname{ci}(x). \tag{19.8.10}$$

The demonstration of (19.8.10) follows along the same lines as the calculation in (19.8.8), and so this completes the proof. □

Entry 19.8.3 (p. 256). *The function $\mathcal{S}(x)$ defined in (19.8.1) has local maxima at $x=(2n+1)\pi$, $n\geq 0$, with the maximum values being*

$$\mathcal{S}(2n+1)=\frac{\pi}{2}+\int_0^\infty \frac{e^{-(2n+1)\pi t}}{1+t^2}dt, \tag{19.8.11}$$

while the local minima are at $x=2n\pi$, $n\geq 1$, with the minimum values being

$$\mathcal{S}(2n)=\frac{\pi}{2}-\int_0^\infty \frac{e^{-2n\pi t}}{1+t^2}dt. \tag{19.8.12}$$

Proof. From elementary calculus, it is trivial to see that the critical points of $\mathcal{S}(x)$ are at $x=n\pi$, $n>0$, when x is positive. Furthermore, it is easy to see that when n is odd, a local maximum is reached, and when n is even, a local minimum is obtained. Furthermore, from (19.8.6) and (19.8.3),

$$\begin{aligned}\mathcal{S}(2n+1)=\frac{\pi}{2}-r\cos((2n+1)\pi-\theta)&=\frac{\pi}{2}+r\cos\theta\\ &=\frac{\pi}{2}+\int_0^\infty \frac{e^{-(2n+1)\pi t}}{1+t^2}dt,\end{aligned}$$

and so (19.8.11) is established. Similarly, (19.8.6) and (19.8.3) immediately yield (19.8.12). □

19.9 Two Infinite Products

Entry 19.9.1 (p. 370). *If $|\operatorname{Re}\beta|<1$, $|\operatorname{Im}\alpha|<1$, and*

$$\cosh\left(\tfrac{1}{2}\pi\beta\right)=\sec\left(\tfrac{1}{2}\pi\alpha\right), \tag{19.9.1}$$

then

$$\prod_{n=0}^\infty\left(\frac{(2n+1)^2+\alpha^2}{(2n+1)^2-\beta^2}\right)^{(-1)^n(2n+1)}=e^{\frac{1}{2}\pi\alpha\beta}. \tag{19.9.2}$$

With the roles of α and β reversed, Entry 19.9.1 is identical to equation (17) in Ramanujan's paper [250], [267, p. 41]. See also (17.2.13) of the present volume. In fact, in place of the condition (19.9.1), Ramanujan wrote the hypothesis

$$\frac{\pi\alpha}{2} = gd\left(\frac{\pi\beta}{2}\right).$$

(Possibly, *gd* denotes the Gudermannian function.) Since Ramanujan only sketched a proof of (19.9.2) in [250], the editors of [267, pp. 336–337] supplied a more detailed proof. An equivalent form of Entry 19.9.1 can be found on page 286 in Ramanujan's second notebook [268], and a proof of Entry 19.9.1 in this form can be found in Berndt's book [41, p. 461, Entry 30]. Lastly, Ramanujan also submitted Entry 19.9.1 as a problem to the *Journal of the Indian Mathematical Society* [248].

Entry 19.9.2 (p. 370 (incorrect)). *For* $|x| < 1$,

$$\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n^2}\right)^{n^2} e^{-x}\right\} = e^{\frac{1}{2}x}, \tag{19.9.3}$$

provided that

$$x = \left\{\frac{1}{\pi}\log\left(\frac{1+\sqrt{5}}{2}\right)\right\}^2. \tag{19.9.4}$$

If we take the logarithm of both sides of (19.9.3), employ the Maclaurin series for $\log(1-z)$, and interchange the order of summation, we deduce that

$$\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{j}\zeta(2j-2)x^j = \frac{x}{2}, \tag{19.9.5}$$

where $\zeta(s)$ denotes the Riemann zeta function. Since $\zeta(0) = -\frac{1}{2}$ [306, p. 19], we can rewrite (19.9.5) in the form

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\zeta(2j)x^j = 0. \tag{19.9.6}$$

Hence, combining (19.9.6) with (19.9.4), we see that Ramanujan claimed that a root of (19.9.6) is (19.9.4), which, if true, would be a remarkable result.

Unfortunately, Entry 19.9.2 is incorrect. This entry also appears in Ramanujan's third notebook [268, p. 365], and in [41, pp. 488–490] Berndt showed that Ramanujan's claim in Entry 19.9.2 is false. In particular, Ramanujan also claimed on the same page in [268] that for $|x| < 1$,

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\zeta(2j)x^{2j+2} = -\frac{1}{\pi^2}\int_0^{\pi x} t^2\coth t\,dt. \tag{19.9.7}$$

If we set

$$x = \frac{1}{\pi}\log\left(\frac{1+\sqrt{5}}{2}\right)$$

above, Ramanujan's claim in Entry 19.9.2 would be equivalent to asserting that the integral on the right side of (19.9.7) equals 0, which is obviously untrue.

19.10 Two Formulas from the Theory of Elliptic Functions

We recall some needed notation from the theory of elliptic functions [39, Chaps. 17, 18, in particular, pp. 101–102]. The incomplete elliptic integral of the first kind is defined, for $0 < \varphi \leq \frac{1}{2}\pi$, by

$$\int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{19.10.1}$$

where k, $0 < k < 1$, is the modulus. The complementary modulus k' is defined by $k' = \sqrt{1-k^2}$. For brevity, we set $x = k^2$. The complete elliptic integral of the first kind is given by (19.10.1) when $\varphi = \frac{1}{2}\pi$ and is denoted by $K = K(k)$. Define $K' := K'(k) := K(k')$. Then in the theory of elliptic functions, we set

$$q := \exp\left(-\pi\frac{K'}{K}\right) =: e^{-y}. \tag{19.10.2}$$

Define, for $0 < \theta \leq \frac{1}{2}\pi$,

$$\theta = \frac{1}{z}\int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

Incomplete elliptic integrals satisfy Jacobi's imaginary transformation. If $0 < \varphi < \frac{1}{2}\pi$, then

$$\int_0^{i\log(\tan(\pi/4+\varphi/2))} \frac{dt}{\sqrt{1-x\sin^2 t}} = i\int_0^{\varphi} \frac{dt}{\sqrt{1-(1-x)\sin^2 t}}. \tag{19.10.3}$$

Entry 19.10.1 (p. 346). *Set, in the notation above,*

$$\frac{2K\theta}{\pi} = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}. \tag{19.10.4}$$

Then

$$\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + 4\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}\sin\{(2n+1)\theta\}}{(2n+1)(1-q^{2n+1})} = \log\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \tag{19.10.5}$$

Entry 19.10.1 coincides with Entry 16(v) of Chap. 18 of Ramanujan's second notebook [268], and a proof can be found in [39, p. 175].

Entry 19.10.2 (p. 346). *In addition to the notation set above, also put*

$$\frac{2K\theta'}{\pi} = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k'^2 \sin^2 t}}. \tag{19.10.6}$$

Then,

$$\theta' + 2\sum_{n=1}^{\infty} \frac{q^n \sinh(2n\theta')}{n(1 + q^{2n})} = \log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \tag{19.10.7}$$

Proof. In the notation above, in particular (19.10.2), in Entry 15(iv) in Chap. 18 of his second notebook [268], Ramanujan claims that

$$\theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n\cosh(ny)} = \varphi; \tag{19.10.8}$$

see [39, pp. 172–173] for a proof. Now in the notation of (19.10.4) and (19.10.6), we will restate (19.10.3) in greater detail, inserting the arguments of the functions θ and θ'. To that end,

$$\theta\left(i\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) = i\,\theta'(\varphi). \tag{19.10.9}$$

Next, in (19.10.8), we substitute (19.10.9) in the form $\theta = i\theta'$. Keeping in mind that φ is defined by (19.10.6), we see from (19.10.9) that we must also replace φ with $i\log\tan(\pi/4 + \varphi/2)$. Hence,

$$i\,\theta' + \sum_{n=1}^{\infty} \frac{\sin(2ni\theta')}{n\cosh(ny)} = i\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \tag{19.10.10}$$

Dividing both sides of (19.10.10) by i and recalling from (19.10.2) that $q = e^{-y}$, we deduce (19.10.7). ∎

Slightly more complicated proofs of the two preceding entries were given by the authors in Part II [13, pp. 238–240].

20

Elementary Results

20.1 Introduction

In this chapter we collect several claims from [269] that are elementary in nature.

20.2 Solutions of Certain Systems of Equations

At the bottom of page 340, there is a short note, "2 pp. of algebraical oddities," which was probably written by G.H. Hardy. On page 341, Ramanujan constructs families of solutions to Euler's Diophantine equation $A^3 + B^3 = C^3 + D^3$, which we discuss in Chap. 8, and so it seems doubtful that page 341 is the second page to which Hardy refers. It seems more likely that Hardy's comment refers to pages 340 and 344, which we discuss in the next section. The last several pages of [269] have been numbered by an unknown person, and in particular, pages 340 and 344 have the numbers 81 and 85 attached to them. It is possible that the pages were shuffled between the times when Hardy recorded his remark and when an anonymous cataloguer tagged the pages with numbers.

The first and third entries on page 340 are in the spirit of the third problem [242] that Ramanujan submitted to the *Journal of the Indian Mathematical Society* and the third article that he published [244] in the same journal.

Entry 20.2.1 (p. 340). *Suppose that*

$$(x^6 + ax)^5 - (x^6 + bx)^5 = A(x^5 + p)^5 + B(x^5 + q)^5 + C(x^5 + r)^5. \quad (20.2.1)$$

Furthermore, write

$$\frac{A}{1 - pz} + \frac{B}{1 - qz} + \frac{C}{1 - rz} = \frac{\alpha + \beta z + \gamma z^2}{1 + \delta z + \epsilon z^2 + \phi z^3}. \quad (20.2.2)$$

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Then

$$\alpha = 5(a-b), \quad \beta = -2\frac{a-b}{a+b}(4a^2+3ab+4b^2), \quad \gamma = 2(a^3-b^3), \tag{20.2.3}$$

$$\delta = -2\frac{a^2+ab+b^2}{a+b}, \ \ \epsilon = a^2+ab+b^2, \ \ \phi = -\frac{a^4+6a^3b+6a^2b^2+6ab^3+b^4}{10(a+b)}. \tag{20.2.4}$$

Proof. Expanding both sides of (20.2.1) by the binomial theorem, we easily find that

$$x^5\sum_{k=0}^{4}\binom{5}{k}(a^{5-k}-b^{5-k})x^{5k} = \sum_{k=0}^{5}\binom{5}{k}(Ap^{5-k}+Bq^{5-k}+Cr^{5-k})x^{5k}.$$

Equating coefficients above, we readily deduce that

$$\begin{cases} Ap^5+Bq^5+Cr^5 = 0, \\ Ap^4+Bq^4+Cr^4 = \frac{1}{5}(a^5-b^5), \\ Ap^3+Bq^3+Cr^3 = \frac{1}{2}(a^4-b^4), \\ Ap^2+Bq^2+Cr^2 = a^3-b^3, \\ Ap+Bq+Cr = 2(a^2-b^2), \\ A+B+C = 5(a-b). \end{cases} \tag{20.2.5}$$

On the other hand, from (20.2.2),

$$(1+\delta z+\epsilon z^2+\phi z^3)\sum_{n=0}^{\infty}(Ap^n+Bq^n+Cr^n)z^n = \alpha+\beta z+\gamma z^2. \tag{20.2.6}$$

Equating constant terms in (20.2.6) and using (20.2.5), we deduce that

$$\alpha = 5(a-b). \tag{20.2.7}$$

Equating coefficients of z^k, $1 \leq k \leq 5$, in (20.2.6), using (20.2.7), and employing the identities in (20.2.5), we find that, respectively,

$$2(a^2-b^2)+5(a-b)\delta = \beta, \tag{20.2.8}$$

$$(a^3-b^3)+2\delta(a^2-b^2)+5\epsilon(a-b) = \gamma, \tag{20.2.9}$$

$$\tfrac{1}{2}(a^4-b^4)+\delta(a^3-b^3)+2\epsilon(a^2-b^2)+5\phi(a-b) = 0, \tag{20.2.10}$$

$$\tfrac{1}{5}(a^5-b^5)+\tfrac{1}{2}\delta(a^4-b^4)+\epsilon(a^3-b^3)+2\phi(a^2-b^2) = 0, \tag{20.2.11}$$

$$\tfrac{1}{5}\delta(a^5-b^5)+\tfrac{1}{2}\epsilon(a^4-b^4)+\phi(a^3-b^3) = 0. \tag{20.2.12}$$

We now observe that (20.2.10)–(20.2.12) are a set of three linear equations in the unknowns δ, ϵ, and ϕ. If we solve this system, we indeed obtain the three proffered values for δ, ϵ, and ϕ given in (20.2.4). Next, we calculate γ from (20.2.9), and we deduce Ramanujan's claimed value for γ in (20.2.3). Lastly, it is easily checked that Ramanujan's value of β in (20.2.3) follows readily from (20.2.8). ☐

Entry 20.2.2 (p. 340). *If*

$$z = \frac{1}{N} \qquad \text{and} \qquad N = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)M, \tag{20.2.13}$$

and if δ, ϵ, and ϕ are given in (20.2.4)*, then*

$$1 + \delta z + \epsilon z^2 + \phi z^3 = 0 \tag{20.2.14}$$

is equivalent to

$$5(a+b)(M^3 - M) = (a-b)(5M^2 - 1). \tag{20.2.15}$$

Proof. Using the values of δ, ϵ, and ϕ from (20.2.4) and then the parameterizations (20.2.13), we find that (20.2.14) can be written as

$$\begin{aligned}
N^3 &- 2\frac{a^2+ab+b^2}{a+b}N^2 + (a^2+ab+b^2)N - \frac{a^4+6a^3b+6a^2b^2+6ab^3+b^4}{10(a+b)} \\
&= \left\{\frac{1}{2}(a+b) + \frac{1}{2}(a-b)M\right\}^3 - 2\frac{a^2+ab+b^2}{a+b}\left\{\frac{1}{2}(a+b) + \frac{1}{2}(a-b)M\right\}^2 \\
&\quad + (a^2+ab+b^2)\left\{\frac{1}{2}(a+b) + \frac{1}{2}(a-b)M\right\} - \frac{a^4+6a^3b+6a^2b^2+6ab^3+b^4}{10(a+b)} \\
&= \frac{1}{10(a+b)}\left\{\frac{5}{4}(a+b)(a-b)^3M^3 - \frac{5}{4}(a-b)^4M^2\right. \\
&\qquad \left. -\frac{5}{4}(a+b)(a-b)^3M + \frac{1}{4}(a-b)^4\right\} = 0.
\end{aligned} \tag{20.2.16}$$

Upon multiplying both sides of (20.2.16) by $40(a+b)/(a-b)^3$, we arrive at

$$5(a+b)M^3 - 5(a-b)M^2 - 5(a+b)M + (a-b) = 0,$$

which is equivalent to (20.2.15). ☐

Entry 20.2.3 (p. 340). *Suppose that*

$$x\left\{(x+a)^3 + (x+b)^3\right\} = A(x+p)^4 + B(x+q)^4 + Cx^4. \tag{20.2.17}$$

Then

$$p = \frac{a^3+b^3}{a^2+b^2-(a-b)\sqrt{3ab}}, \qquad q = \frac{a^3+b^3}{a^2+b^2+(a-b)\sqrt{3ab}}, \tag{20.2.18}$$

$$A = -\frac{(a^2+b^2-(a-b)\sqrt{3ab})^4}{8(a-b)\sqrt{3ab}(a^3+b^3)^2}, \quad B = \frac{(a^2+b^2+(a-b)\sqrt{3ab})^4}{8(a-b)\sqrt{3ab}(a^3+b^3)^2}, \tag{20.2.19}$$

$$C = \frac{(a^3-b^3)(a-b)^3}{(a^3+b^3)^2}. \tag{20.2.20}$$

Proof. Expanding both sides of (20.2.17) by the binomial theorem, we see that

$$x\sum_{k=0}^{3}\binom{3}{k}(a^{3-k}+b^{3-k})x^k = \sum_{k=0}^{4}\binom{4}{k}(Ap^{4-k}+Bq^{4-k})x^k + Cx^4.$$

Equating coefficients of x^k, $0 \le k \le 4$, we find that

$$\begin{cases} 0 = Ap^4 + Bq^4, \\ \frac{1}{4}(a^3+b^3) = Ap^3 + Bq^3, \\ \frac{1}{2}(a^2+b^2) = Ap^2 + Bq^2, \\ \frac{3}{4}(a+b) = Ap + Bq, \\ 2 = A + B + C. \end{cases} \tag{20.2.21}$$

For brevity, set

$$a_1 = 2, \quad a_2 = \frac{3}{4}(a+b), \quad a_3 = \frac{1}{2}(a^2+b^2), \quad a_4 = \frac{1}{4}(a^3+b^3), \quad a_5 = 0. \tag{20.2.22}$$

We now employ a clever idea of Ramanujan [244], [267, pp. 18–19]. Write

$$\phi(\theta) := \frac{A}{1-\theta p} + \frac{B}{1-\theta q} + C = \sum_{n=1}^{\infty} a_n\theta^{n-1} = \frac{A_1 + A_2\theta + A_3\theta^2}{1 + B_1\theta + B_2\theta^2}, \tag{20.2.23}$$

where, by expanding the left side in geometric series, we see that indeed $a_1, a_2, \ldots, a_5$ are given by (20.2.22), and where A_1, A_2, A_3 and B_1, B_2 are constants that we now proceed to determine. Rewriting the last equality in (20.2.23) in the form

$$(1 + B_1\theta + B_2\theta^2)(a_1 + a_2\theta + a_3\theta^2 + a_4\theta^3 + a_5\theta^4 + \cdots) = A_1 + A_2\theta + A_3\theta^2,$$

and equating coefficients on both sides, we find that

$$\begin{cases} A_1 = a_1, \\ A_2 = a_2 + a_1 B_1, \\ A_3 = a_3 + a_2 B_1 + a_1 B_2, \\ 0 = a_4 + a_3 B_1 + a_2 B_2, \\ 0 = a_5 + a_4 B_1 + a_3 B_2. \end{cases} \tag{20.2.24}$$

Using (20.2.22), we see that the last two equations in (20.2.24) can be written in the form

$$\frac{1}{2}(a^2+b^2)B_1 + \frac{3}{4}(a+b)B_2 = -\frac{1}{4}(a^3+b^3),$$
$$\frac{1}{4}(a^3+b^3)B_1 + \frac{1}{2}(a^2+b^2)B_2 = 0.$$

Solving simultaneously this pair of linear equations, we find that

$$B_1 = -\frac{2(a^2+b^2)(a^3+b^3)}{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)}, \tag{20.2.25}$$

$$B_2 = \frac{(a^3+b^3)^2}{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)}. \tag{20.2.26}$$

Using (20.2.25) and (20.2.26), we can determine A_3 from the third equality of (20.2.24). Accordingly,

$$A_3 = \frac{2(a^2+b^2)^3 - 3(a+b)(a^2+b^2)(a^3+b^3) + 2(a^3+b^3)^2}{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)}. \tag{20.2.27}$$

We now use (20.2.25) in the second equality of (20.2.24) to conclude that

$$A_2 = \frac{12(a+b)(a^2+b^2)^2 - 9(a+b)^2(a^3+b^3) - 16(a^2+b^2)(a^3+b^3)}{4\{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)\}}. \tag{20.2.28}$$

Now that we have determined A_1, A_2, A_3 and B_1, B_2, we return to (20.2.23) and expand the rational function on the far right-hand side into partial fractions,

$$\begin{aligned} \phi(\theta) &= \frac{A_1 + A_2\theta + A_3\theta^2}{1 + B_1\theta + B_2\theta^2} = \frac{p_1}{1-\theta q_1} + \frac{p_2}{1-\theta q_2} + p_3 \\ &= \frac{p_1+p_2+p_3 - (p_1q_2 + p_2q_1 + q_1p_3 + q_2p_3)\theta + q_1q_2p_3\theta^2}{1 - (q_1+q_2)\theta + q_1q_2\theta^2}. \end{aligned} \tag{20.2.29}$$

But we also see from (20.2.23) that $p_1 = A$, $p_2 = B$, $p_3 = C$, $q_1 = p$, and $q_2 = q$. Using these observations and comparing coefficients in the two representations for the same rational function in (20.2.29), we find that

$$\begin{cases} A_1 = p_1 + p_2 + p_3 = A + B + C, \\ A_2 = -(p_1q_2 + p_2q_1 + q_1p_3 + q_2p_3) = -Aq - Bp - C(p+q), \\ A_3 = p_3q_1q_2 = Cpq, \\ B_1 = -(q_1 + q_2) = -p - q, \\ B_2 = q_1q_2 = pq. \end{cases} \tag{20.2.30}$$

We now are ready to determine p, q, C, and A, B in this order. From the last two equations of (20.2.30) and from (20.2.25) and (20.2.26), we see that

$$-\frac{2(a^2+b^2)(a^3+b^3)}{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)} = -p - \frac{1}{p}\,\frac{(a^3+b^3)^2}{4(a^2+b^2)^2 - 3(a+b)(a^3+b^3)}.$$

Solving this equation, we find that

$$p = \frac{a^3+b^3}{a^2+b^2-(a-b)\sqrt{3ab}},$$

as claimed in (20.2.18). Then, from either of the last two equalities in (20.2.30), we readily compute that

$$q = \frac{a^3+b^3}{a^2+b^2+(a-b)\sqrt{3ab}}.$$

Alternatively (and more easily), we could simply verify that the given values of p and q simultaneously solve the last two equations of (20.2.30). Having found p and q, we turn to the third equation in (20.2.30) to determine C. After a moderate amount of elementary algebra, we find that C is given by (20.2.20). Lastly, we employ the first two equations in (20.2.30) to demonstrate that A and B are given by (20.2.19). Admittedly, a heavy amount of tedious, but straightforward, elementary algebra is necessary. □

M.D. Hirschhorn [162] has devised a somewhat different approach to the three identities at the beginning of page 340.

In the last entry on page 340, Ramanujan attempts to find a family of solutions to the diophantine equation

$$A^4 + B^4 + C^4 = D^4 + E^4 + F^4. \tag{20.2.31}$$

On page 384 of his third notebook [269], Ramanujan provides two families of solutions to (20.2.31). See [40, pp. 94–95, 106–107] for a discussion of Ramanujan's solutions. Unfortunately, Ramanujan's recorded family of solutions for (20.2.31) is erroneous. Hirschhorn and L. Vaserstein independently (and almost simultaneously) found a correct version of Ramanujan's formula. The factors $(n^2+3)(n^4+42n^2+9)$ were inadvertently omitted by Ramanujan from the last two terms on the right-hand side below.

Entry 20.2.4 (p. 340; corrected). *A family of solutions to* (20.2.31) *is given by*

$$\begin{aligned}
&\left\{x^5(n^2+3)^2(n^4+42n^2+9)^2+x(n^2-3)(n^2+6n+3)\right\}^4 \\
&\quad -\left\{x^5(n^2+3)^2(n^4+42n^2+9)^2+x(n^2-3)(n^2-6n+3)\right\}^4 \\
&= \left\{x^4(n^2+3)(n^4+42n^2+9)(n^4+6n^3+18n^2-18n+9)+(n^2-3)\right\}^4 \\
&\quad -\left\{x^4(n^2+3)(n^4+42n^2+9)(n^4-6n^3+18n^2+18n+9)+(n^2-3)\right\}^4 \\
&\quad +\left\{6nx^4(n^2+3)(n^4+42n^2+9)(n^2+4n-3)\right\}^4 \\
&\quad -\left\{6nx^4(n^2+3)(n^4+42n^2+9)(n^2-4n-3)\right\}^4 .
\end{aligned}$$

20.3 Radicals

Most of page 344 in [269] is devoted to eight identities involving, on one side, a quotient of binomial conjugates and, on the other side, geometric type series in the variable g. In each case, there is a condition, such as $g^5 = 2$, attached.

Entry 20.3.1 (p. 344). *If* $g^4 = 5$, *then*

$$\frac{\sqrt[5]{3+2g}-\sqrt[5]{4-4g}}{\sqrt[5]{3+2g}+\sqrt[5]{4-4g}} = 2+g+g^2+g^3. \tag{20.3.1}$$

(The symbol g is almost completely obliterated in [269].) A clever proof of Entry 20.3.1 was constructed by Hirschhorn [163] using the elementary and easily proved principle of *componendo et dividendo* [19, p. 320], which we now describe. Suppose that $a \neq b$ and $c \neq d$. Then

$$\frac{a}{b} = \frac{c}{d}$$

if and only if

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

First Proof of Entry 20.3.1. Hirschhorn begins his proof of Entry 20.3.1 with the trivial observation that (20.3.1) is equivalent to the identity

$$\frac{\sqrt[5]{3+2g}+\sqrt[5]{4g-4}}{\sqrt[5]{3+2g}-\sqrt[5]{4g-4}} = \frac{2+g+g^2+g^3}{1}. \tag{20.3.2}$$

By the principle of *componendo et dividendo*, with $a = \sqrt[5]{3+2g}$, $b = \sqrt[5]{4g-4}$, $2c = 3+g+g^2+g^3$, and $2d = 1+g+g^2+g^3$, we deduce that (20.3.2) is equivalent to the identity

$$\frac{\sqrt[5]{3+2g}}{\sqrt[5]{4g-4}} = \frac{3+g+g^2+g^3}{1+g+g^2+g^3},$$

which in turn is equivalent to the identity

$$\frac{3+2g}{4g-4} = \left(\frac{3+g+g^2+g^3}{1+g+g^2+g^3}\right)^5. \tag{20.3.3}$$

Now observe that

$$(4x-4)(3+x+x^2+x^3)^5-(3+2x)(1+x+x^2+x^3)^5 = (x^4-5)P(x), \tag{20.3.4}$$

where $P(x)$ is a certain polynomial of degree 12. Setting $x = g$ in (20.3.4) and using the hypothesis $g^4 = 5$, we complete the proof of (20.3.3) and hence also of the proof of Entry 20.3.1. □

Second Proof of Entry 20.3.1. We provide another proof, which was communicated to the authors by M. Somos [291]. We begin with the easily verified identity

$$8(3+2x) = (x-1)(x+1)^5 - (x^4-5)(x^2+4x+5).$$

Setting $x = g$, recalling that $g^4 = 5$, and multiplying both sides by 4, we obtain the identity

$$2^5(3+2g) = (4g-4)(g+1)^5.$$

Taking the fifth root of both sides and introducing the abbreviations $a = (3+2g)^{1/5}$ and $b = (4g-4)^{1/5}$, we have

$$\frac{a}{b} = \frac{g+1}{2}, \quad \text{or} \quad \frac{a-b}{b} = \frac{g-1}{2}. \tag{20.3.5}$$

Now $g^4 = 5$ or $g^4 - 1 = 4$, which we write in the factored form

$$(g-1)(1+g+g^2+g^3) = 4,$$

or

$$\frac{4}{g-1} = 1+g+g^2+g^3. \tag{20.3.6}$$

From (20.3.5) and (20.3.6), we find that

$$\frac{2b}{a-b} = 1+g+g^2+g^3.$$

Adding 1 to both sides yields

$$\frac{a+b}{a-b} = 2+g+g^2+g^3,$$

which is the same as (20.3.1). □

We now state the remaining seven identities. Each of the first three can be proved using the principle of *componendo et dividendo*; each of the seven can be readily verified by rationalizing the denominator (if necessary) on the left-hand side, multiplying the right-hand side by the rationalized denominator, simplifying with the use of the given condition, solving for the nth root, raising both sides to the nth power, and then simplifying once again with the use of the auxiliary condition. We provide a proof of one of the identities using the first method and a proof of one of the remaining identities using the second method. Since the other identities can be proved in the same fashions, we leave the remaining proofs as exercises. Although proofs are easily given, the following fundamental question remains: How did Ramanujan discern these identities? We have been unable to answer this obvious question.

Entry 20.3.2 (p. 344). *We have*

$$g^5=2, \qquad \frac{\sqrt{g+3}+\sqrt{5g-5}}{\sqrt{g+3}-\sqrt{5g-5}}=g+g^2, \tag{20.3.7}$$

$$g^5=2, \qquad \frac{\sqrt{g^2+1}+\sqrt{4g-3}}{\sqrt{g^2+1}-\sqrt{4g-3}}=\frac{1}{5}\left(1+g^2+g^3+g^9\right)^2, \tag{20.3.8}$$

$$g^5=3 \qquad \frac{\sqrt{g^2+1}+\sqrt{5g-5}}{\sqrt{g^2+1}-\sqrt{5g-5}}=\frac{1}{g}+g+g^2+g^3,$$

$$g^5=2, \qquad \sqrt{1+g^2}=\frac{g^4+g^3+g-1}{\sqrt{5}},$$

$$g^5=2, \qquad \sqrt{4g-3}=\frac{g^9+g^7-g^6-1}{\sqrt{5}},$$

$$g^5=3, \qquad \sqrt[3]{2-g^3}=\frac{1+g-g^2}{\sqrt[3]{5}},$$

$$g^5=2, \qquad \sqrt[5]{1+g+g^3}=\frac{\sqrt{1+g^2}}{\sqrt[10]{5}}.$$

To the right of the penultimate identity above, Ramanujan writes $g=3$, and to the right of the last identity above, Ramanujan writes $g^5+5g^3+5g+2=0$.

First Proof of (20.3.7). Apply the principle of *componendo et dividendo* with $a=\sqrt{g+3}$, $b=\sqrt{5g-5}$, $2c=1+g+g^2$, and $2d=-1+g+g^2$. Hence, (20.3.7) is equivalent to the identity

$$\frac{\sqrt{g+3}}{\sqrt{5g-5}}=\frac{1+g+g^2}{-1+g+g^2},$$

which in turn is equivalent to

$$\frac{g+3}{5g-5}=\left(\frac{1+g+g^2}{-1+g+g^2}\right)^2. \tag{20.3.9}$$

We now observe that

$$(x+3)(-1+x+x^2)^2 - 5(x-1)(1+x+x^2)^2 = -4(x^5-2). \qquad (20.3.10)$$

Setting $x = g$ in (20.3.10) and recalling that $g^5 = 2$, we complete the proof. □

Second Proof of (20.3.7). For a second proof, we offer M. Somos's [291] variation of the first proof. We first observe that

$$(x+3)(x^2+x-1)^2 = x^5 + 5x^4 + 5x^3 - 5x^2 - 5x + 3. \qquad (20.3.11)$$

In order to obtain terms on the right-hand side that are all multiples of 5, we add $4(x^5-2)$ to both sides above to deduce that

$$\begin{aligned}(x+3)(x^2+x-1)^2 + 4(x^5-2) &= 5(x^5+x^4+x^3-x^2-x-1)\\ &= 5(x^2+x+1)^2(x-1). \qquad (20.3.12)\end{aligned}$$

Thus, from (20.3.11) and (20.3.12), we deduce the identity

$$5(x-1)(x^2+x+1)^2 = (x+3)(x^2+x-1)^2 + 4(x^5-2).$$

Substituting $x = g := 2^{1/5}$ above, we arrive at

$$5(g-1)(g^2+g+1)^2 = (g+3)(g^2+g-1)^2.$$

Rearrange this identity so that we can apply the principle of *componendo et dividendo* with $a = \sqrt{g+3}$, $b = \sqrt{5g-5}$, $c = g^2+g+1$, and $d = g^2+g-1$. The identity (20.3.7) now follows. □

Proof of (20.3.8). Rationalizing the denominator on the left-hand side of (20.3.8), we find that

$$\frac{\sqrt{g^2+1}+\sqrt{4g-3}}{\sqrt{g^2+1}-\sqrt{4g-3}} = \frac{g^2+4g-2+2\sqrt{(g^2+1)(4g-3)}}{g^2-4g+4}. \qquad (20.3.13)$$

In view of (20.3.8), we thus wish to examine, with the use of the condition $g^5 = 2$,

$$\begin{aligned}\frac{1}{5}\left(1+g^2+g^3+g^9\right)^2(g^2-4g+4) &= (1+2g+2g^2+2g^3+g^4)(g^2-4g+4)\\ &= 6g+g^2+2g^3-2g^4. \qquad (20.3.14)\end{aligned}$$

Hence, from (20.3.8), (20.3.13), and (20.3.14), it suffices to show that

$$\sqrt{(g^2+1)(4g-3)} = 1+g+g^3-g^4. \qquad (20.3.15)$$

Squaring both sides of (20.3.15) and using the condition $g^5 = 2$ to simplify, we easily establish the truth of (20.3.15). □

Lastly, Somos [291] provided the following explanation for the addenda accompanying the last two entries in Entry 20.3.2. For the first, note that

$$(2-g^3)-\tfrac{1}{5}(1+g-g^2)^3=\tfrac{1}{5}(3-g)(3-g^5).$$

Thus, $g=3$ is a root of the left-hand side. For the second, note that

$$(1+g+g^3)^2-\tfrac{1}{5}(1+g^2)^5=\tfrac{1}{5}(2-g^5)(2+5g+5g^3+g^5).$$

Thus, when $g^5+5g^3+5g+2=0$, we obtain the last equality of Entry 20.3.2.

20.4 More Radicals

At the bottom of page 344 in [269], Ramanujan offers four entries involving equalities of radical expressions.

Entry 20.4.1 (p. 344). *We have*

$$\sqrt[3]{\frac{1}{3}}+\sqrt[3]{\frac{5}{3}}=\sqrt{\frac{\sqrt[3]{5}-1}{2-\sqrt[3]{5}}\sqrt[3]{3}}=\sqrt[3]{\frac{3+\sqrt[3]{5}}{\sqrt[3]{5}-1}}=\sqrt[5]{\frac{3\sqrt[3]{3}+\sqrt[3]{15}}{2-\sqrt[3]{5}}}. \tag{20.4.1}$$

If a, b, and c are arbitrary numbers, then

$$\begin{aligned}\left\{\sqrt[3]{(a+b)(a^2+b^2)}-a\right\}&\left\{\sqrt[3]{(a+b)(a^2+b^2)}-b\right\}\\&=\frac{\sqrt[3]{(a+b)^2}-\sqrt[3]{a^2+b^2}}{\sqrt[3]{(a+b)^2}+\sqrt[3]{a^2+b^2}}(a^2+ab+b^2),\end{aligned} \tag{20.4.2}$$

$$\frac{(\sqrt{a^2+ab+b^2}-a)(\sqrt{a^2+ab+b^2}-b)}{a+b-\sqrt{a^2+ab+b^2}}=a+b, \tag{20.4.3}$$

$$\begin{aligned}&\left\{-a+\sqrt{(c+a)(a+b)}\right\}\left\{-b+\sqrt{(a+b)(b+c)}\right\}\left\{-c+\sqrt{(b+c)(c+a)}\right\}\\&\qquad=2\left(\frac{ab+bc+ca}{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}\right)^2.\end{aligned} \tag{20.4.4}$$

Proof. We provide a proof by Somos [291] of the first equality of (20.4.1), which was incorrectly written by Ramanujan, who forgot the factor $\sqrt[3]{3}$ under the radical sign on the right-hand side. We emphasize that it is a simple matter to verify each of the equalities in (20.4.1). We content ourselves with offering only brief discussions of the remaining identities of Entry 20.4.1, since only elementary algebra is involved.

Following Somos [291], we consider

$$(x-2)(x+1)^2 = x^3 - 3x - 2,$$

and add $3(x-1)$ to both sides to make the coefficient of x equal to zero, to wit,

$$(x-2)(x+1)^2 + 3(x-1) = x^3 - 5.$$

Substituting $x = g := 5^{1/3}$, we find that

$$(g-2)(g+1)^2 = 3(1-g), \quad \text{or} \quad (g+1)^2 = \frac{3(g-1)}{2-g}.$$

Taking the square root of each side, we have

$$1 + g = \sqrt{\frac{3(g-1)}{2-g}}.$$

Dividing both sides by $3^{1/3}$, we obtain the first equality of (20.4.1).

The identity (20.4.2) is a beautiful identity, which is not difficult to verify by crossmultiplication. However, this is clearly not how Ramanujan discovered it. More insight is needed.

The identity (20.4.3) can be checked by crossmultiplication in a matter of seconds.

The last identity (20.4.4) is exquisite. In [269], Ramanujan expressed the right-hand side of (20.4.4) in terms of the reciprocal of the quotient. It appears that one would need computer algebra to check (20.4.4), but on crossmultiplication, we see that there are only four different kinds of terms to check, and so the use of symmetry substantially shortens the task with therefore no computer algebra needed. Hirschhorn has devised a clever proof of (20.4.4) by setting $a+b = 4C^2$, $b+c = 4A^2$, and $c+a = 4B^2$. Then it is easily checked that both sides are equal to $8(A-B+C)^2(A+B-C)^2(A-B-C)^2$. But still, a more natural proof of (20.4.4) is desired. □

20.5 Powers of 2

Page 345 in [269] is devoted to a single table, with no explanation for it. All of the entries in the table are of the form

$$2^{31} \prod_{j \in \{1,2,4,8,16\}} \left(1 + \frac{1}{2^j}\right).$$

We augment the table with an additional column to the right providing the decimal representation of the product on the left. An examination of these numbers reveals that Ramanujan has listed the numbers in decreasing order, which, with a little thought, is also clear from an inspection of Ramanujan's table.

$$
\begin{aligned}
&2^{32} &&= 4{,}294{,}967{,}296\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 4{,}294{,}967{,}295\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8}) &&= 4{,}294{,}901{,}760\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^{16}}) &&= 4{,}278{,}255{,}360\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4}) &&= 4{,}278{,}190{,}080\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 4{,}042{,}322{,}160\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^8}) &&= 4{,}042{,}260{,}480\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^{16}}) &&= 4{,}026{,}593{,}280\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^2}) &&= 4{,}026{,}531{,}840\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 3{,}435{,}973{,}836\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8}) &&= 3{,}435{,}921{,}408\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^{16}}) &&= 3{,}422{,}604{,}288\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^4}) &&= 3{,}422{,}552{,}064\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 3{,}233{,}857{,}728\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^8}) &&= 3{,}233{,}808{,}384\\
&2^{31}(1+\tfrac{1}{2})(1+\tfrac{1}{2^{16}}) &&= 3{,}221{,}274{,}624\\
&2^{31}(1+\tfrac{1}{2}) &&= 3{,}221{,}225{,}472\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 2{,}863{,}311{,}530\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8}) &&= 2{,}863{,}267{,}840\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^{16}}) &&= 2{,}852{,}170{,}240\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^4}) &&= 2{,}852{,}126{,}720\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 2{,}694{,}881{,}440\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^8}) &&= 2{,}694{,}840{,}320\\
&2^{31}(1+\tfrac{1}{2^2})(1+\tfrac{1}{2^{16}}) &&= 2{,}684{,}395{,}520\\
&2^{31}(1+\tfrac{1}{2^2}) &&= 2{,}684{,}354{,}560\\
&2^{31}(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 2{,}290{,}649{,}224\\
&2^{31}(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^8}) &&= 2{,}290{,}614{,}272\\
&2^{31}(1+\tfrac{1}{2^4})(1+\tfrac{1}{2^{16}}) &&= 2{,}281{,}736{,}192\\
&2^{31}(1+\tfrac{1}{2^4}) &&= 2{,}281{,}701{,}376\\
&2^{31}(1+\tfrac{1}{2^8})(1+\tfrac{1}{2^{16}}) &&= 2{,}155{,}905{,}152
\end{aligned}
$$

$$2^{31}(1 + \tfrac{1}{2^8}) = 2{,}155{,}872{,}256$$
$$2^{31}(1 + \tfrac{1}{2^{16}}) = 2{,}147{,}516{,}416$$
$$2^{31} = 2{,}147{,}483{,}648$$

20.6 An Elementary Approximation to π

Entry 20.6.1 (p. 370).

$$\frac{9}{5} + \sqrt{\frac{9}{5}} = 3.14164\cdots = \pi + 0.00005\ldots.$$

The truth of this approximation to π is easily checked. We do not know how Ramanujan discovered it. However, M. Somos noted that

$$\frac{6}{5}\left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{9}{5} + \sqrt{\frac{9}{5}}.$$

Thus, Ramanujan might have taken π, divided it by the square of the golden ratio, and observed that it was close to $\frac{6}{5}$.

21

A Strange, Enigmatic Partial Manuscript

21.1 Introduction

The partial manuscript on pages 257 and 258 of [269] is an assorted collection of claims, many of which are wrong and, at times, outrageous. In fact, some claims are so flagrant that one has to conclude that Ramanujan must have intended something different from what he wrote. We might also conjecture that this manuscript arises from his teenage years, as he was just beginning to think about theoretical analysis and analytic number theory. We quote Ramanujan throughout, with our remarks put in square brackets, as is our custom.

21.2 A Strange Manuscript

All variables considered in the following results are *positive.* $a_1 + a_2 + \cdots + a_n$ can be expressed in terms of n with an error of $O(\max a_n - \min a_n)$. Perhaps we may go even as far as ascertaining the maximum and the minimum of O and it is not possible to go beyond that. For example,

$$\sum_{k=1}^{n} d(k) = n(2\gamma - 1 + \log n) + O(\max d(n)) \tag{21.2.1}$$

$$= n(2\gamma - 1 + \log n) + O\left(2^{O\left(\frac{\log n}{\log\log n}\right)}\right). \tag{21.2.2}$$

[Clearly, some conditions must be placed on the sequence $\{a_n\}$ in order for Ramanujan's opening statement to have validity. If we assume the truth of (21.2.1), then (21.2.2) follows from a result of Ramanujan [251, Eq. (20)], [267, p. 46]. However, by a famous theorem of G.H. Hardy [145], the error term equals $\Omega(n^{1/4})$, which is incompatible with (21.2.1).]

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$\log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \cdots$ to n terms can be expressed in terms of n with an error of $o(1)$ or even $O(n^{-1/2+\epsilon})$; for

$$O(\max \log p_n - \min \log p_n) + O(n^{-1/2+\epsilon}).$$

[Here p_n denotes the nth prime. These last claims are obviously wrong. Ramanujan must have possessed some (unorthodox) interpretation for his claims, but we are clueless about his intent.]

$$\text{The maximum order of } a_n = \max \sqrt[r]{(\text{average order of } a_n^r)} \tag{21.2.3}$$

for the variable r. For example,

(i) $\max d(n)$ can be shown to be $2^{\frac{\log n}{\log\log n}(1+\epsilon)}$,

(ii) The maximum order of $\pi(p) = \operatorname{Li}(p) + \max \sqrt[r]{\text{average order of } (p-p')^r}$, where p' is the prime just less than the prime p.

[As previously indicated, the assertion (i) is correct. By the prime number theorem, the average order of $p - p'$ is $\log p$. However, then Ramanujan's statement (ii) reads

$$\text{The maximum order of } \pi(p) = \operatorname{Li}(p) + \log p,$$

which is incorrect by the prime number theorem. Thus, Ramanujan may have had a different meaning of *average order* than is customarily accepted today.]

If b_n is steadily increasing and if $\max a_n > b_n$, and if

$$\sum_{k=1}^{\infty} a_k e^{-kx} - \sum_{k=1}^{\infty} b_k e^{-kx} = O(1) \tag{21.2.4}$$

as $x \to 0$, then

$$\sum_{k=1}^{m} a_k e^{-kx} - \sum_{k=1}^{m} b_k e^{-kx} = O(\max a_n - \min a_n). \tag{21.2.5}$$

If in (21.2.5) a_n also is steadily increasing, then

$$a_n \sim b_n. \tag{21.2.6}$$

[Ramanujan evidently uses *steadily increasing* to mean *strictly monotonically increasing*. The hypothesis $\max a_n > b_n$ is unclear. Evidently, Ramanujan means that $\max a_n$ is larger than b_n for *all* n, $1 \leq n < \infty$. The conclusion (21.2.5) is not generally valid. For example, suppose that $a_k \equiv 1$ and $b_k = 1 - 1/k^2$, $k \geq 1$. Then the hypothesis $\max a_n > b_n$ is satisfied. Moreover,

$$\sum_{k=1}^{m} (a_k - b_k) e^{-kx} \to \sum_{k=1}^{m} \frac{1}{k^2} \neq \max a_n - \min a_n = 0.$$

In regard to the hypotheses (21.2.4), we can apply a Tauberian theorem found in Hardy's book *Divergent Series* [148, p. 153, Theorem 89] to conclude the following corollary.

Corollary 21.2.1. *If*

$$S(x) := \sum_{k=1}^{\infty} (a_k - b_k) e^{-kx}$$

is convergent for $x > 0$, $S(x) \to s$ *as* $x \to 0$, *and*

$$a_k - b_k = o\left(\frac{1}{k}\right),$$

as $k \to \infty$, *then*

$$\sum_{k=1}^{\infty} (a_k - b_k)$$

converges to s.

Since hypotheses on a_n and b_n are given at the beginning of the claim, it is difficult to ascertain what is meant by a *conclusion* about a_n and b_n in the asymptotic formula (21.2.6), which is the last statement on page 257. On page 258, there are three paragraphs numbered 5–7. On page 257, there is a faint 1 before the third paragraph, and so evidently some pages of this partial manuscript are missing. Ramanujan indicates that the two-line statement under paragraph 5 is to be moved to the end of paragraph 6, and we have done that. The next three statements are designated (i)–(iii) and do not seem to have any connection with the other claims on these two pages.]

6. If a_n and b_n are steadily increasing and if

$$\sum_{n=1}^{\infty} \frac{1}{a_n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{b_n^s}$$

are both convergent when $s > k$ and both are divergent when $s = k$ and if the difference between the two series be $O(1)$ when $s = k$, then

$$a_n \sim b_n. \tag{21.2.7}$$

5. Analogous results in case of

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

[Harold Diamond has kindly shown us that the conclusions in assertions numbered 6 and 5 are incorrect. We define two Dirichlet series that converge absolutely for $\operatorname{Re} s > 1$. Define, for N even,

$$a_n = 2^N, \qquad 2^N \leq n < 2^{N+1},$$
$$b_n = \frac{1}{n}, \qquad 2^N \leq n < 2^{N+1},$$

while for odd N, define

$$b_n = 2^N, \qquad 2^N \le n < 2^{N+1},$$
$$a_n = \frac{1}{n}, \qquad 2^N \le n < 2^{N+1}.$$

We note that if N is even,

$$\sum_{n=2^N}^{2^{N+1}-1} \frac{1}{a_n} = \sum_{n=2^N}^{2^{N+1}-1} \frac{1}{2^N} = 1$$

and

$$\sum_{n=2^N}^{2^{N+1}-1} \frac{1}{b_n} = \sum_{n=2^N}^{2^{N+1}-1} \frac{1}{n} \sim \log 2, \quad N \to \infty.$$

If N is odd, similar formulas exist, but with the roles of a_n and b_n reversed. Clearly, all of the hypotheses of Assertions 5 and 6 hold. However, it is also clear that $a_n \nsim b_n$ as $n \to \infty$, contradicting Ramanujan's claim.]

7. If a_n and b_n are steadily increasing and if

$$\sum_{n=1}^{\infty} e^{-a_n x} \sim \sum_{n=1}^{\infty} e^{-b_n x} \tag{21.2.8}$$

as $x \to 0$,

[The conclusion of the claim in paragraph 7 is not provided, since the next page in Ramanujan's partial manuscript is missing.]

(i) The number of numbers of the form $2^m 3^n$ such as $1, 2, 3, 4, 6, 8, 9, 12, \ldots$ less than x

$$= \frac{1}{2} \cdot \frac{\log(2x)}{\log 2} \cdot \frac{\log(3x)}{\log 3} + O(1). \tag{21.2.9}$$

(ii) The number of numbers of the form $a^2 + b^2$ less than x

$$= C \int \frac{dx}{\sqrt{\log x}} + O(x^{1/2+\epsilon}), \tag{21.2.10}$$

where

$$C = \sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)} \times \sqrt{\left(1-\frac{1}{23^2}\right)\left(1-\frac{1}{31^2}\right)\left(1-\frac{1}{43^2}\right)\cdots}. \tag{21.2.11}$$

$$\sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)} = \left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right). \quad (21.2.12)$$

(iii) The number of numbers of the form a^2b^3 such as $1, 4, 8, 9, 16, 25, 27, 32, 36, \ldots$ less than x

$$= \sqrt{4.723034x} - \sqrt[3]{3.10227x} + O(x^{1/6+\epsilon}). \quad (21.2.13)$$

[Ramanujan's assertion (i) is incorrect as it stands. However, if interpreted as an asymptotic formula, the claim is correct and can be found in his first letter to Hardy [267, p. xxiv], [64, p. 23]. It is a special case of the more general problem of finding an asymptotic formula for those numbers $\leq x$ of the form $a^m b^n$, with a and b fixed. This problem is thoroughly examined by Hardy in his book [147, Chap. 5]. It is also mentioned by Ramanujan on page 309 in his second notebook [268], [40, pp. 66–67]. A thorough discussion of (i) (in corrected form) is given in [40, pp. 62–69].

The claim (21.2.10) with the incorrect error term has a long history dating back to Ramanujan's first letter to Hardy [147, pp. xxiv, xxviii]. Strangely, the constant given in (21.2.11) is also incorrect; the correct constant is the reciprocal of that given in (21.2.11). The claim (21.2.10) is also found on page 307 in Ramanujan's second notebook [268]. For detailed discussions of (21.2.10) and correct versions, see Hardy's book [147, pp. 60–63] and Berndt's book [40, pp. 60–62].

The curious identity (21.2.12) was observed by Ramanujan before departing for England. It is found twice in his notebooks, on page 309 in the second notebook and page 363 in the third notebook [268]. A brief discussion of (21.2.12) can be found in [40, p. 20].

The constants multiplying $\sqrt{x}$ and $\sqrt[3]{x}$ in (21.2.13) are difficult to interpret in [269]. Unless we are gravely misreading or misinterpreting them, they are incorrect. In fact, on page 324 in his earlier second notebook, Ramanujan gives the correct result [40, p. 73]

$$\sum_{a^2b^3\leq x} 1 = \zeta\left(\frac{3}{2}\right)\sqrt{x} + \zeta\left(\frac{2}{3}\right)\sqrt[3]{x} + O(x^{1/5})$$
$$= 2.6123753\sqrt{x} - 3.6009377\sqrt[3]{x} + O(x^{1/5}).$$

Although there is some suspicion that he had a *bona fide* proof, Ramanujan's error term in (21.2.13) is superior to that given in his second notebook. But whether he had a proof or not, the error term is indeed correct. For example, H.-E. Richert [272] has established an error term of $O(x^{2/15})$.

More generally, E. Landau [207], [208, p. 24] proved the following elementary result.

Theorem 21.2.1. *Let α and β be fixed positive numbers such that $\alpha \neq \beta$. Then*

$$\sum_{a^\alpha b^\beta \leq x} 1 = \zeta\left(\frac{\beta}{\alpha}\right) x^{1/\alpha} + \zeta\left(\frac{\alpha}{\beta}\right) x^{1/\beta} + \Delta(\alpha, \beta; x),$$

where $\Delta(\alpha, \beta; x) = O(x^{1/(\alpha+\beta)})$, as $x \to \infty$.

The exact order of the error term $\Delta(\alpha, \beta; x)$ is not known, but improvements on Landau's initial theorem can be found in Richert's paper [272] and E. Krätzel's book [200, pp. 221–227].]

Location Guide

For each page of Ramanujan's lost notebook on which we have discussed or proved entries in this book, we provide below a list of those chapters, sections, or entries in which these pages are discussed.

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